Long Term Growth Model (LTGM v4.3) - Model Description

Steven Pennings (spennings@worldbank.org)

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- **NEW** in LTGM v4: effect of growth on poverty (via log-normal income distribution, Section 5)
- The neoclassical growth model is based on Solow (1956), Swan (1956) and Hevia and Loayza (2012)
  - There are only two key parts: the production function and capital accumulation.
- Model 1: assume a path for the investment share of GDP (I/Y) $\rightarrow$ implied per-capita GDP growth.
- Model 2: assume a path of growth in GDP per capita $\rightarrow$ required investment share of GDP (I/Y).
- Model 3: assume a path for the savings share of GDP (S/Y) $\rightarrow$ implied per-capita GDP growth.
- A Current Account Balance or External Debt constraint converts savings (S/Y) into (I/Y) in Model 3. The constraint also allows savings to be calculated as residual in Model 1 and 2 (see Section 3 for details).
- Section **4** summarizes the drivers of per-capita GDP growth in one equation (and compares to the ICOR).

1 Model 1: Growth given investment

1.1 The production function

I assume a standard production function where Y_t is GDP, A_t is the total factor productivity, K_t is the capital stock, and $h_t L_t$ is *effective* labor used in production, which can be further decomposed as h_t human capital *per worker*, and L_t is the number of workers. β is the labor share.

$$Y_t = A_t K_t^{1-\beta} (h_t L_t)^\beta \tag{1}$$

N_t is the total population and so the number of workers can be decomposed into $L_t = \varrho_t \omega_t N_t$ where ϱ_t is the participation rate (labor force/working age population) and ω_t is the working age population-to-total population ratio.[1]

We can divide by N_t or L_t to get all variables in per capita or per worker terms. In terms of notation, the *default is per worker*, but we add "PC" to denote per capita (i.e y_t is output per worker and y_t^{pc} is output per capita). k_t is capital per worker. h_t is already in per worker terms terms.

$$y_t^{pc} \equiv \frac{Y_t}{N_t} = \frac{Y_t}{L_t} \varrho_t \omega_t = y_t \varrho_t \omega_t$$

$$y_t \equiv \frac{Y_t}{L_t} = A_t k_t^{1-\beta} h_t^\beta$$

$$y_t^{pc} = \varrho_t \omega_t y_t = A_t \varrho_t \omega_t k_t^{1-\beta} h_t^\beta$$

From this equation we can calculate growth rates from t to t+1:

$$\frac{y_{t+1}}{y_t} = \frac{A_{t+1} k_{t+1}^{1-\beta} h_{t+1}^\beta}{A_t k_t^{1-\beta} h_t^\beta} = \frac{A_{t+1}}{A_t} \left[\frac{k_{t+1}}{k_t}\right]^{1-\beta} \left[\frac{h_{t+1}}{h_t}\right]^\beta$$

Rewrite this equation in terms of $g_{y,t+1}$, the growth rate of GDP *per worker* from t to t+1 (eg 0.05).

[1] We assume no unemployment or underemployment, so everyone in the labor force works. In an earlier version of this note (with different notation), I assumed that $\varrho_t = \omega_t = 1$ so that population and the labor force we were the same.

$$1+g_{y,t+1}=(1+g_{A,t+1})\left[1+g_{k,t+1}\right]^{1-\beta}\left[1+g_{h,t+1}\right]^{\beta} \tag{2}$$

In Equation 2, output growth is driven by productivity growth ($g_{A,t+1}$ the growth rate of TFP), capital deepening ($g_{k,t+1}$ is the growth rate is capital *per worker*), and $g_{h,t+1}$ is growth rate of human capital *per worker.*

$$\frac{y^{pc}_{t+1}}{y^{pc}_{t}}=\left[\frac{\varrho_{t+1}}{\varrho_t}\right]\left[\frac{\omega_{t+1}}{\omega_t}\right]\left[\frac{y_{t+1}}{y_t}\right]$$

Growth in output *per capita* is just output *per worker* adjusted for changes in participation and the working age population. Specifically, for a given growth rate of output per worker, output growth can also be driven by a *demographic transition* (growth in the working age to population ratio $g_{\omega,t+1}$), or an *increase in labor force participation* (growth in the participation rate $g_{\varrho,t+1}$).

$$1+g^{pc}_{y,t+1}=\left[1+g_{\omega,t+1}\right]\left[1+g_{\varrho,t+1}\right]\left[1+g_{y,t+1}\right] \tag{3}$$

1.2 Physical Capital accumulation

Equation 4 is capital accumulation, where I_t is investment.

$$K_{t+1}=(1-\delta)K_t+I_t \tag{4}$$

Before progressing further, we need to rewrite labor force growth in terms of growth rates. Substituting $L_t=\varrho_t\omega_tN_t$ we get the following equation, where $g_{N,t+1}$ is population growth between t and t+1.

$$\frac{L_{t+1}}{L_t}=\frac{\varrho_{t+1}\omega_{t+1}N_{t+1}}{\varrho_t\omega_tN_t}=\left\{1+g_{\varrho,t+1}\right\}\left\{1+g_{\omega,t+1}\right\}(1+g_{N,t+1})$$

The next step to is write capital accumulation in *per worker* terms. Start with Equation 4 and divide everything by L_t.

$$\left[\frac{K_{t+1}}{L_{t+1}}\right]\left[\frac{L_{t+1}}{L_t}\right]=(1-\delta)\frac{K_t}{L_t}+\frac{I_t}{L_t}$$

Now write in terms of growth rates and in *per worker* terms.

$$k_{t+1}\left\{1+g_{\varrho,t+1}\right\}\left\{1+g_{\omega,t+1}\right\}(1+g_{N,t+1})=(1-\delta)k_t+i_t$$

We can divide everything by k_t

$$\frac{k_{t+1}}{k_t}(1+g_{N,t+1})\left\{1+g_{\varrho,t+1}\right\}\left\{1+g_{\omega,t+1}\right\}=(1-\delta)+\frac{i_t}{k_t}$$

Then divide and multiply by y_t (output per worker). Where $\frac{I_t}{Y_t}$ is the investment share of GDP and $\frac{K_t}{Y_t}$ is the capital to output ratio.

$$(1+g_{k,t+1})(1+g_{N,t+1})\left\{1+g_{\varrho,t+1}\right\}\left\{1+g_{\omega,t+1}\right\}=(1-\delta)+\frac{i_t}{y_t}\frac{y_t}{k_t}$$

Rearrange to get the growth rate of capital *per worker.*[2]

$$(1+g_{k,t+1})=\frac{(1-\delta)+\frac{I_t}{Y_t}/\frac{K_t}{Y_t}}{(1+g_{N,t+1})(1+g_{\varrho,t+1})(1+g_{\omega,t+1})} \tag{5}$$

Equation 5 determines capital deepening (in per worker terms).

To solve the model we need K_t/Y_t. We take the initial value K_0/Y_0 from the MFMod Database, PWT 8.1 and 9. For later periods we need to update the the capital-to-output ratio in the next period using Equation 6 and the values for $g_{k,t+1}$ and $g_{y,t+1}$ we have calculated in Equation 5 and 2.

$$\frac{K_{t+1}}{Y_{t+1}}=\frac{k_{t+1}}{y_{t+1}}=\frac{k_{t+1}}{k_t}\frac{y_t}{y_{t+1}}\frac{k_t}{y_t}$$

[2]Note that $\frac{I_t}{Y_t}=\frac{i_t}{y_t}$ and $\frac{K_t}{Y_t}=\frac{k_t}{y_t}$ because the L_t in numerator and denominator cancel.

$$\frac{K_{t+1}}{Y_{t+1}} = \frac{(1 + g_{k,t+1})}{(1 + g_{y,t+1})} \frac{K_t}{Y_t} \tag{6}$$

1.3 Steps to solve the model

In order to solve the model, we first need to input some exogenous variables:

- Parameters for β (the labor share), δ (the capital depreciation rate), and $\frac{K_0}{Y_0}$ (initial capital-to-output ratio)
- Assumption of paths for $\left\{g_{A,t}, g_{h,t}, g_{\omega,t}, g_{\varrho,t}, g_{N,t}, \frac{I_t}{Y_t}\right\}_{t=1}^{T}$

Then we can calculate the growth rate of GDP per capita using the following steps.

1. Calculate the growth rate of capital per worker using Equation 5 (using exogenous & predetermined variables)
2. Calculate the growth rate of output *per worker* using Equation 2 (using $g_{k,t+1}$from step 1)
3. Calculate the growth rate of output *per capita* using Equation 3 (using $g_{y,t+1}$from step 2)
4. Update next period's capital to output ratio using Equation 6 (using $g_{k,t+1}$ and $g_{y,t+1}$from steps 1 and 2)

1.4 Parameters and initial values

- Users can choose parameters and initial values from a range of data sources and time periods using the drop-down menus in the LTGM spreadsheet, or simply type in their own values.
  - Missing parameters or initial values are interpolated based on income group averages, which triggers a red warning cell.
- β (the labor share) is taken from PWT 8.1, 9.0, and 9.1 and the GTAP database. 0.4-0.7 are reasonable numbers.
- δ (the capital depreciation rate) and $\frac{K_0}{Y_0}$ (the initial capital-to-output ratio) are taken from PWT 8.1, 9.0, or 9.1.[3]
- $\frac{I_t}{Y_t}$ is the investment share of GDP. The user will have to make an assumption about the future path of $\frac{I_t}{Y_t}$.[4]
- $g_{A,t+1}$ is exogenous total factor productivity growth, and reasonable numbers are 0% (pessimistic), 1% (moderate) or 2% (optimistic). Faster productivity growth can be obtained from (for example) technology adoption, greater competition, reduced regulation, or factors moving from less efficient to more efficient sectors. Historical averages calculated from PWT 8.1. 9.0 and 9.1 (or applying PWT 8.1 methodology with GTAP labor shares)
- $g_{\varrho,t+1}$ is the growth rate of labor participation. Historical data from the ILO or country authorities. In practice, the most important determinant of $g_{\varrho,t+1}$ is female labor force participation.
- $g_{N,t+1}$ is exogenous population growth, and $g_{\omega,t+1}$ is growth in the working age to total population ratio. These are both taken from the World Bank Health Nutrition and Population Statistics: Population estimates and projections (link), with $g_{\omega,t+1}$ determined by the age structure of the population.
- $g_{h,t+1}$ is exogenous human capital per worker growth and taken from PWT 8.1, 9.0 or 9.1. Higher human capital growth, eg from more schooling or more effective schooling, will increase growth.[5]

[3] The K_0/Y_0 ratio is calculated as rkna/rgdpna in PWT 8.1, 9.0 and rnna/rgdpna in 9.1 (national accounts prices).

[4] Historical averages are taken from Fixed Domestic Investment as a share of Real GDP from MFMod (NEGDIFTOTKD /NYGDPMKTPKD) and WDI.

[5] Hevia and Loayza (2012) refer to this as $h_t = e^{\phi(E_t)}$, which is the efficiency of a workers with E_t average years of schooling. If $E_t = 0$, $h_t = 1$, so h_t represents the efficiency of a worker with E_t years of education relative to one with no education. ie. if $h_t = 2$, for a country average years of schooling is twice as productive as a worker with no education. $\phi(E)$ governs the return to an extra year of schooling, and in PWT is piecewise linear, where the marginal return to schooling is 13.4% for the first 4 years, 10.1% for following 4-8 years and 6.8% for years of schooling after that, schooling from Barro and Lee's dataset v1.3 (Inklaar and Timmer 2013, see Equation 15 and 16).

2 Model 2: Calculating the Investment Share of GDP to achieve a given rate of GDP per capita growth

It is straightforward to rearrange the equations above to calculate the investment rate necessary to generate a required rate of per-capita GDP growth (users can also set a poverty target, see Section 5).

First, calculate the growth rate of output *per worker* consistent with desired rate of growth of output *per capita* using Equation 3.

$$1 + g_{y,t+1} = \frac{1 + \bar{g}^{pc}_{y,t+1}}{[1 + g_{\omega,t+1}]\,[1 + g_{\varrho,t+1}]}$$

Next substitute this into Equation 2

$$(1 + g_{A,t+1})\,[1 + g_{k,t+1}]^{1-\beta}\,[1 + g_{h,t+1}]^{\beta} = \frac{1 + \bar{g}^{pc}_{y,t+1}}{[1 + g_{\omega,t+1}]\,[1 + g_{\varrho,t+1}]}$$

Next substitute Equation 5 into Equation 2 to remove the capital growth rate.

$$(1 + g_{A,t+1})\,[1 + g_{h,t+1}]^{\beta}\left[\frac{(1-\delta) + \frac{I_t}{Y_t}/\frac{K_t}{Y_t}}{(1 + g_{N,t+1})(1 + g_{\varrho,t+1})(1 + g_{\omega,t+1})}\right]^{1-\beta} = \frac{1 + \bar{g}^{pc}_{y,t+1}}{[1 + g_{\omega,t+1}]\,[1 + g_{\varrho,t+1}]}$$

Then do some algebra to isolate I/Y on the LHS.

$$\left[(1-\delta) + \frac{I_t}{Y_t}/\frac{K_t}{Y_t}\right]^{1-\beta} = \frac{\left\{1 + \bar{g}^{pc}_{y,t+1}\right\}(1 + g_{N,t+1})^{1-\beta}(1 + g_{\varrho,t+1})^{1-\beta}(1 + g_{\omega,t+1})^{1-\beta}}{(1 + g_{A,t+1})\,[1 + g_{h,t+1}]^{\beta}\,[1 + g_{\omega,t+1}]\,[1 + g_{\varrho,t+1}]}$$

$$\left[(1-\delta) + \frac{I_t}{Y_t}/\frac{K_t}{Y_t}\right]^{1-\beta} = \frac{\left\{1 + \bar{g}^{pc}_{y,t+1}\right\}(1 + g_{N,t+1})^{1-\beta}}{(1 + g_{A,t+1})\,[1 + g_{h,t+1}]^{\beta}\,[1 + g_{\omega,t+1}]^{\beta}\,[1 + g_{\varrho,t+1}]^{\beta}}$$

$$\frac{I_t}{Y_t}/\frac{K_t}{Y_t} = \frac{\left\{1 + \bar{g}^{pc}_{y,t+1}\right\}^{\frac{1}{1-\beta}}(1 + g_{N,t+1})}{(1 + g_{A,t+1})^{\frac{1}{1-\beta}}\,[1 + g_{h,t+1}]^{\frac{\beta}{1-\beta}}\,[1 + g_{\omega,t+1}]^{\frac{\beta}{1-\beta}}\,[1 + g_{\varrho,t+1}]^{\frac{\beta}{1-\beta}}} - (1-\delta)$$

$$\frac{I_t}{Y_t} = \frac{K_t}{Y_t}\left[\frac{\left\{1 + \bar{g}^{pc}_{y,t+1}\right\}^{\frac{1}{1-\beta}}(1 + g_{N,t+1})}{(1 + g_{A,t+1})^{\frac{1}{1-\beta}}\,[1 + g_{h,t+1}]^{\frac{\beta}{1-\beta}}\,[1 + g_{\omega,t+1}]^{\frac{\beta}{1-\beta}}\,[1 + g_{\varrho,t+1}]^{\frac{\beta}{1-\beta}}} - (1-\delta)\right] \tag{7}$$

Given $\{\bar{g}^{pc}_{y,t}\}^{T}_{t=1}$ one can calculate required investment using Equation 7. As before, future values of $\frac{K_t}{Y_t}$ can be updated for period t+1, t+2.. using Equation 6, with the growth rate of capital calculated from Equation 5.

Equation 7 states that required investment is increasing in the desired per capita growth rate $\bar{g}^{pc}_{y,t+1}$, the deprecation rate δ, the population growth rate $g_{N,t+1}$ and also the capital-to-output ratio K_t/Y_t. Growth in productivity ($g_{A,t+1}$), human capital ($g_{h,t+1}$), the working age population ratio ($g_{\omega,t+1}$) and the participation rate ($g_{\varrho,t+1}$) all reduce the required investment rate.

3 The External Balance Constraint and Model 3 (Growth Given Savings)

Savings is converted into investment (and vice versa) using a binding external balance constraint which can either be in the form of a path for (i) CAB_t/Y_t or (ii) the external debt share of GDP D_t/Y_t (and foreign direct investment FDI_t/Y_t).[6]

A Current Account Balance Constraint

The model is simplest assuming a path CAB_t/Y_t.[7]

$$\frac{I_t}{Y_t} = \frac{S_t}{Y_t} - \frac{CAB_t}{Y_t} \tag{8}$$

- **Model 3 (CAB/Y constraint):** Given an assumption for national savings as a share of GDP (S_t/Y_t), simply combine with the CAB_t/Y_t constraint and use Equation 8 to calculate I_t/Y_t Then use Step 1-4 from Model 1 (Section 1.3) to calculate growth.

- **Model 1 and 2 (CAB/Y constraint):** Rearrange 8 and combine the CAB_t/Y_t constraint with path for I_t/Y_t assumed (Model 1) or implied (Model 2) to generate implied savings as: $S_t/Y_t = I_t/Y_t - CAB_t/Y_t$.

An External Debt Constraint

Alternatively, one can assume an external debt constraint, combined with a path for foreign direct investment (FDI_t). From a simplified version balance of payments identities, the CAB_t equals the acquisition of Net Foreign Assets (NFA_t) less the incurrence of Net Foreign Liabilities (NFL_t). Assets and liabilities are recorded end-of-period.

$$CAB_t = \Delta NFA_t - \Delta NFL_t \tag{9}$$

The change in net foreign liabilities can be decomposed into net inflows of foreign direct investment (FDI), as well as the accumulation of total external debt D_t (portfolio liabilities, public and private). For simplicity, we assume no changes in the stock of net foreign assets, which is a benign assumption for most developing countries.

$$\Delta NFL_t = FDI_t + (D_t - D_{t-1}) \qquad \Delta NFA_t \approx 0$$

Substituting into Equation 9, and dividing by GDP (Y_t) and using $Y_t/Y_{t-1} = (1+g^{pc}_{y,t})(1+g_{N,t})$, one can write the CAB_t/Y_t as:

$$\frac{CAB_t}{Y_t} = -\left[\frac{D_t}{Y_t} - \frac{D_{t-1}/Y_{t-1}}{(1+g^{pc}_{y,t})(1+g_{N,t})}\right] - \frac{FDI_t}{Y_t} \tag{10}$$

Combining Equations 8 and 10, one can relate savings and investment with an external debt constraint.

$$\frac{I_t}{Y_t} = \frac{S_t}{Y_t} + \frac{FDI_t}{Y_t} + \left[\frac{D_t}{Y_t} - \frac{D_{t-1}/Y_{t-1}}{(1+g^{pc}_{y,t})(1+g_{N,t})}\right] \tag{11}$$

This can also be re written in terms of required savings.

$$\frac{S_t}{Y_t} = \left[\frac{I_t}{Y_t} - \frac{FDI_t}{Y_t}\right] - \left[\frac{D_t}{Y_t} - \frac{D_{t-1}/Y_{t-1}}{(1+g^{pc}_{y,t})(1+g_{N,t})}\right] \tag{12}$$

- **Model 3 (D/Y constraint):** Assume a path for external debt-to-GDP (D_t/Y_t), and net inflows of Foreign Debt Investment FDI_t/Y_t, and use Equation 11 to calculate I_t/Y_t for a given level of savings (S_t/Y_t). Then use Step 1-4 from Model 1 (Section 1.3) to calculate growth.

- **Model 1 and 2 (D/Y constraint):** Combine paths for D_t/Y_t and FDI_t/Y_t with path for I_t/Y_t assumed (Model 1) or implied (Model 2) to generate implied savings using Equation 12.

The effect of FDI on required national savings: With an external debt constraint, an increase in FDI_t/Y_t acts as a substitute for national savings (S_t/Y_t). That is, in Equation 12, national savings only have to cover the fraction of investment not funded by FDI $[I_t/Y_t - FDI_t/Y_t]$, rather than the whole amount.

[6]Initial values of CAB_t/Y_t are taken from the MFMod Database and WDI. The World Development Indicators are the source of FDI_t/Y_t (code: BX.KLT.DINV.WD.GD.ZS.) and D_t/Y_t (calculated as DT.DOD.DECT.CD ÷ NY.GDP.MKTP.CD).

[7]For economies not open to capital flows, assume $CAB_t/Y_t \approx 0$.

4 Understanding the drivers of growth

To understand the drivers of growth, in this section we take log-linear approximation to simplify the formulas. Note that quantitative analysis should be done using the exact equations above because even small approximation errors can compound over time. First combine Equation 2 and 3:

$$1 + g^{pc}_{y,t+1} = [1 + g_{\omega,t+1}] [1 + g_{\varrho,t+1}] (1 + g_{A,t+1}) [1 + g_{k,t+1}]^{1-\beta} [1 + g_{h,t+1}]^{\beta}$$

Taking logs, and using the approximation $ln(1 + x) \approx x$ (for small x) this becomes

$$g^{pc}_{y,t+1} \approx g_{A,t+1} + g_{\omega,t+1} + g_{\varrho,t+1} + (1 - \beta)g_{k,t+1} + \beta g_{h,t+1} \tag{13}$$

Taking logs of the capital-per-worker growth Equation 5, yields:

$$ln(1 + g_{k,t+1}) = ln \left[1 + \frac{I_t}{Y_t} / \frac{K_t}{Y_t} - \delta\right] - ln(1 + g_{N,t+1}) - ln(1 + g_{\varrho,t+1}) - ln(1 + g_{\omega,t+1})$$

Applying the $ln(1 + x) \approx x$ approximation (for small x):

$$g_{k,t+1} \approx \frac{I_t}{Y_t} / \frac{K_t}{Y_t} - \delta - g_{N,t+1} - g_{\varrho,t+1} - g_{\omega,t+1} \tag{14}$$

Combining Equation 13 and 14 gives the approximate determinants of growth:

$$g^{pc}_{y,t+1} \approx g_{A,t+1} + \beta(g_{\omega,t+1} + g_{\varrho,t+1} + g_{h,t+1}) + (1 - \beta) \left[\frac{I_t}{Y_t} / \frac{K_t}{Y_t} - \delta - g_{N,t+1}\right] \tag{15}$$

The effect of most factors (except TFP) on growth depends on the labor share β, which is around 0.5 on average across all countries (PWT 8.1, 9.0 and 9.1).[8]

- TFP growth ($g_{A,t+1}$) has the largest direct effect on growth: a 1ppt increase in TFP growth increases growth by 1ppt (regardless of β)
- A 1ppt increase human capital growth ($g_{h,t+1}$), labor force participation rate growth ($g_{\varrho,t+1}$) or working age population share growth ($g_{\omega,t+1}$) increase per capita GDP growth by βppt. If $\beta \approx 0.5$, than a 1ppt increase in each of these factors, has *half* the effect as a 1ppt increase in TFP growth.
- Population growth ($g_{N,t+1}$) and depreciation (δ) reduce per capita GDP growth by $1 - \beta$, because they reduce capital depth (capital per worker) by either reducing the amount of capital (δ) or increasing the number of workers ($g_{N,t+1}$).
- The effect of an increase in the the investment share of GDP depends on both the capital share $(1 - \beta)$, as well as the existing capital-to-output ratio (K/Y). Assuming $1 - \beta = 0.5$, a large 10ppt increase in the investment share of GDP (eg from 20% to 30%), raises growth by 2.5ppt per year if $K/Y = 2$ (i.e $0.1 \times 0.5/2$), but only 1.25ppt if $K/Y = 4$.
  - *This means that a investment-led growth strategy will quickly become less effective, unless it is accompanied by other reforms to boost productivity, human capital or participation to mitigate the increase in K/Y.*

[8] In standard growth account exercises for OECD countries like the US, the labor share is around 2/3.

4.1 Relation to the Incremental Capital to Output Ratio (ICOR)

Many countries use the Incremental Capital to Output Ratio (ICOR) to analyze the effectiveness of investment in boosting growth. Specifically, the (gross) marginal ICOR is the percentage point increase in the investment share of GDP needed to boost headline GDP growth by 1%.[9]

$$g_Y \approx \frac{1}{ICOR}\frac{I}{Y} \tag{16}$$

where $g_{Y,t+1} \equiv \dfrac{Y_{t+1}-Y_t}{Y_t} \approx g^{pc}_{y,t+1} + g_{N,t+1}$ is the growth rate of headline GDP (not per capita). As the relationship in Equation 16 is assumed to be proportional, the *average* ICOR ($ICOR_a$) is equal to the *marginal* ICOR ($ICOR_m$) and so is just called the *ICOR*. Take two examples : (a) if the *marginal* ICOR is 4.3, then if the country wants to increase headline GDP growth by 1%, it must increase the investment share of GDP by 4.3 ppt; (b) if the *average* ICOR is 4.3, an 8% target GDP growth rate can be achieved with an investment share of 34.4% of GDP ($I/Y = ICOR_g \times g_Y$ =4.3 × 8% = 34.4%, rearranging Equation 16).

In the long-term growth model (LTGM), the approximate relationship between GDP growth and I/Y is linear but not proportional, which means that the ICOR only applies to an analysis of *extra* units of growth or investment.[10] Rearranging Equation 15:

$$g_{Y,t+t} \approx \underbrace{\beta g_{N,t+1} + g_{A,t+1} + \beta(g_{\omega,t+1} + g_{\varrho,t+1} + g_{h,t+1}) - (1-\beta)\delta}_{Intercept \quad which \quad doesn't \quad depend \quad on \quad I/Y} + \underbrace{\frac{1-\beta}{K_t/Y_t}}_{1/ICOR_m} \times \frac{I_t}{Y_t} \tag{17}$$

In the LTGM, to boost GDP growth by 1%, one must raise investment share of GDP by:

$$ICOR^{LTGM}_{m,t} = \frac{1}{1-\beta}\frac{K_t}{Y_t} \tag{18}$$

For example, suppose $\beta = 0.5$, and $K_0/Y_0 = 2.2$ then the *marginal* ICOR is 4.4, so one needs to increase the investment share of GDP by 4.4ppts to boost headline GDP growth by 1%.

Critically, the marginal ICOR is *not* constant over time. A investment-led growth strategy not accompanied by growth in TFP etc will rapidly increase K/Y, leading to an *increase* in the ICOR. This makes future investment less effective in boosting growth. The level of growth, and the rate of K/Y growth also depend on the terms in the intercept.

Readers will note that the inverse of the marginal ICOR is just the *marginal product of capital (MPK)*:

$$MPK \equiv \frac{\partial Y_t}{\partial K_t} = (1-\beta)\frac{Y_t}{K_t} = \frac{1}{ICOR_{m,t}}.$$

The net return on capital is $R = MPK - \delta$. In the example above, if the marginal ICOR is 4.4, then the marginal product of capital is $4.4^{-1} = 23\%$. With $\delta = 5\%$ the net return on capital is 23%-5%=18% per year. A rising marginal ICOR as K/Y increases during an investment-led growth program is the same as saying that the return to capital is falling.

[9]As the name suggests, the Incremental Capital to Output ratio originally referred to net investment: $\dfrac{\Delta K_t}{\Delta Y_t} = \dfrac{I^{Net}_t/Y_{t-1}}{\Delta Y_t/Y_{t-1}} = \dfrac{(I_t - \delta K_{t-1})/Y_t}{\Delta Y/Y_t}$. However, in practice net investment was hard to measure in developing countries due to a lack of data on depreciation rates and capital stocks. As result many analysts approximated net investment with gross investment, giving the rule-of-thumb measure presented in Equation 16. It should be noted that the rule-of-thumb gross ICOR is not the same as the original net ICOR: the net ICOR will be *smaller* by $\dfrac{\delta K_{t-1}/Y_t}{\Delta Y/Y_t}$, which could be important if growth rates are small. For example, with $K/Y = 2$ and $\delta = 0.05$, the net ICOR is smaller by 5 if GDP growth is 2%, but only by 2 is GDP growth is 5%

[10]That is, the *marginal* ICOR and the *average* ICOR will be different. The average ICOR will change with the investment rate.

5 Poverty Extension: the effect of growth on the poverty rate

The poverty rate is determined by the *distribution* of per capita income, as well as its average level. LTGM v4 introduces the income distribution by assuming that the natural log of income per capita follows a normal distribution, $ln(y^{pc}) \sim N(\mu, \sigma^2)$.[11] The log-normal income distribution greatly simplifies the calculation of poverty rates, needs very few parameters and little data and empirically is good approximation of the majority of the income distribution (see Lopez and Serven 2006 and Bourguignon 2007). The addition of the poverty extension does not substantially affect the workings of the Model 1-3 above.

The poverty headcount rate P is the proportion of people with incomes below the poverty line L. Combined with the mean μ and standard deviation σ of the *underlying* normal distribution (not the mean/SD the actual income distribution), the poverty rate can be calculated from the standard normal cumulative density function (CDF) as in Equation 19.[12]

$$P_t = \Phi(\frac{lnL - \mu_t}{\sigma_t}) \tag{19}$$

For a log normal distribution, the Gini coefficient of income inequality G, is a transformation of standard deviation σ of the *underlying* normal distribution:

$$G_t = 2\Phi(\frac{\sigma_t}{\sqrt{2}}) - 1 \tag{20}$$

To calculate μ from data on P and L invert Equation 19:

$$\mu_t = lnL - \sigma_t \Phi^{-1}(P_t) \tag{21}$$

where σ can be calculate from G by inverting Equation and 20:

$$\sigma_t = \sqrt{2} \times \Phi^{-1}(\frac{G_t + 1}{2}) \tag{22}$$

The mean income (GDP per capita) is given by $exp(\mu + \sigma^2/2)$. Absent any change in income inequality (i.e. with constant G and σ), economic growth shifts the whole income distribution to the right (proportionately) by increasing μ. Allowing for a change in inequality, the per capita growth rate is given by Equation 23:

$$\begin{aligned} 1 + g^{pc}_{y,t+1} &= \bar{y}^{pc}_{t+1}/\bar{y}^{pc}_t = \frac{exp(\mu_{t+1} + \sigma^2_{t+1}/2)}{exp(\mu_t + \sigma^2_t/2)} \\ &= exp(\mu_{t+1} - \mu_t + \frac{1}{2}(\sigma^2_{t+1} - \sigma^2_t)) \end{aligned} \tag{23}$$

We can rewrite this as Equation 24, which is used update the mean of the underlying normal distribution in Models 1 and 3.

$$\mu_{t+1} = ln(1 + g^{pc}_{y,t+1}) + \mu_t - \frac{1}{2}(\sigma^2_{t+1} - \sigma^2_t) \tag{24}$$

Using the approximation $ln(1+g) \approx g$ (for small g), this becomes: $\mu_{t+1} \approx g^{pc}_{y,t+1} + \mu_t - {}^1\!/_2(\sigma^2_{t+1} - \sigma^2_t)$ which suggests that with constant income inequality ($\sigma^2_{t+1} = \sigma^2_t$), an extra percentage point of per capita GDP growth increases μ by one percentage point.

Steps to solve for poverty rates in Model 1, 2 and 3 using the log-normal distribution

In models 1 and 3, growth fundamentals (investment savings etc) determine the path of per capita growth $\{g^{pc}_{y,t+1}\}$, from which we calculate the change in the poverty rate. The steps are as follows:

1. Assume a path for the Gini coefficient on income $\{G_t\}$ from the first period until the end of the simulation (this could be constant), and then calculate σ_t for each year using Equation 22.[13]

[11] Thanks to Aart Kraay for suggesting this approach. Although the distribution of income is always log normal, μ and σ vary across countries and over time.

[12] The CDF (proportion less than x) is $Pr(X \leq x) = \Phi(x)$, which is *normsdist(x)* in Excel. The inverse function $\Phi^{-1}(Pr)$, is *normsinv(Pr)* in Excel.

[13] This can be done in Excel as $\sigma_t = sqrt(2) * NORMSINV(0.5 * (G_t + 1))$

2. Choose an initial poverty line L and initial headcount poverty rate P_t at that poverty line. The pre-loaded defaults are the international extreme poverty line ($1.90/day 2011 PPP), and the extreme poverty rate for the most recent household survey in PovcalNet. Alternatively, users can manually enter their own $\{P, L\}$ for the own national poverty line (which is often quite different).[14] Calculate the initial μ_t using Equation 21. [15]

3. For each period after the first, update μ_{t+1} using Equation 24.

4. For each period after the first, calculate the poverty rate P_{t+1}using Equation 19 (given μ_{t+1}, σ_{t+1} and L).[16]

Growth Elasticity of Poverty (GEP) — understanding the effect of growth on poverty

In the literature, the *growth elasticity of poverty (GEP)* ε_p is the percentage (not percentage point) fall in the headcount poverty rate from a 1% increase in per capita income (i.e. from a 1% per capita growth rate). For a log normal distribution, the GEP is given by Equation 25, which helps the LTGM user to compare their estimates with those in the empirical literature.[17] Bourguignon (2007) reports empirical estimates of about 1.5 across poverty spells, though earlier papers estimate the GEP of 2 or even 3.

However, the GEP varies substantially across countries and over time. With a log normal income distribution, the GEP is mechanically higher for countries with *low* rates of poverty, because even a small change in the poverty headcount is large as a percentage of a very small base. The GEP is also higher in countries that are more equal (a smaller Gini coefficient of income). As noted by Bourguignon (2007) this means that a reduction in inequality has a "double dividend": first it reduces poverty in and of itself, and second it increases the growth elasticity of poverty.

$$\varepsilon_{p,t} \equiv -\frac{\partial lnP_t}{\partial ln\bar{y}_t} = -\frac{\partial P_t}{\partial \mu_t}\frac{1}{P_t} = \frac{1}{\sigma}\frac{\phi(\frac{lnL-\mu_t}{\sigma_t})}{\Phi(\frac{lnL-\mu_t}{\sigma_t})} \tag{25}$$

A closely related metric is *growth semi-elasticity of poverty*, which we define as the *percentage point* change in poverty for an extra 1% increase in per capita income (i.e. from a 1% per capita growth rate) as in Equation 26.

$$\Delta_p \equiv -\frac{\partial P_t}{\partial ln\bar{y}_t} = \varepsilon_{p,t} \times P_t = \frac{1}{\sigma}\phi(\frac{lnL-\mu_t}{\sigma_t}) \tag{26}$$

The growth semi-elasticity of poverty is inverse-U shaped in the poverty rate, with the largest response of poverty rates in *percentage points* generally occurring in countries with a poverty rate of around 0.5. At this point there are many people just below the poverty line who can be moved out of poverty by a small increase in income. The *growth semi-elasticity of poverty* is calculated as a memorandum item in the LTGM spreadsheets.

GEP Implementation in the LTGM Spreadsheet In the LTGM, users can input their own GEP instead of using the one implied by the log–normal income distribution (Equation 25). For Model 1, Model 3, and Model 2 (with a non-poverty target), the poverty rate is given calculated by:

$$P_{t+1} = (1 - \varepsilon_{p,t+1} \times g^{pc}_{y,t+1})P_t \tag{27}$$

Alternatively, in Model 2 (with a poverty target) required growth is given by:

$$g^{pc}_{y,t+1} = -(P_{t+1}/P_t - 1)/\varepsilon_{p,t+1} \tag{28}$$

[14]It is only the initial poverty rate P that affects the model. Changes in L (for example changing from per-day to per month, or the currency of measurement), do not affect the model as μscales accordingly.

[15]This can be done in Excel as $\mu_{i,t} = ln(L_i) - \sigma_{i,t}NORMSINV(P_{i,t})$

[16]This can be done in Excel as $P_{t+1} = NORMSDIST((ln(L) - \mu_{t+1})/\sigma_{t+1})$

[17]An increase in average income (keeping σ constant) always reduces the poverty rate, so we follow Bourguignon (2007) and make the elasticity positive by pre-multiplying by -1. The second equality in Equation 25 comes from the mean of a log-normal distribution $ln\bar{y}_i = \mu_i + \sigma_i^2/2$ (keeping σ_i constant) and the third equality comes from applying Leibniz's rule to Equation 19. This equation is similar to Equation 3′ in Bourguignon (2007). Here $\phi(.)$ is the normal probability distribution function (in Excel $NORMDIST(x, 0, 1, FALSE)$ and Equation 25 is $(1/\sigma_{t+1}) * NORMDIST((ln(L) - \mu_{t+1})/\sigma_{t+1}, 0, 1, FALSE)/NORMSDIST((ln(L) - \mu_{t+1})/\sigma_{t+1})$.

Shared Prosperity Premium (SPP)

One of the goals of the World Bank is to "Promote shared prosperity by fostering the income growth of the bottom 40% for every country."

In the LTGM poverty extension, the average income of the bottom 40% of the population is given by Equation 29, where $k_t \equiv exp(\sigma_t \Phi^{-1}(0.4) + \mu_t)$ is the income cutoff that defines the bottom 40% (which changes over time and across countries).[18]

$$E(y^{pc}|y^{pc} < k_t) = 0.4^{-1} e^{\mu+\sigma^2/2} \Phi(\Phi^{-1}(0.4) - \sigma_t) \tag{29}$$

The income share of the bottom 40% of the population ($SB40$) can be expressed as[19]

$$\begin{aligned} SB40_t &\equiv \frac{E(y^{pc}|y^{pc} < k_t) \times 0.4}{\bar{y}_t^{pc}} \\ &= \frac{0.4^{-1} e^{\mu_t+\sigma_t^2/2} \Phi(\Phi^{-1}(0.4) - \sigma_t) \times 0.4}{e^{\mu_t+\sigma_t^2/2}} \\ &= \Phi(\Phi^{-1}(0.4) - \sigma_t) \end{aligned} \tag{30}$$

In terms of the Gini coefficient (using Equation 22) the income share of the bottom 40% can be written as:

$$SB40_t = \Phi(\Phi^{-1}(0.4) - \sqrt{2} \times \Phi^{-1}((G_t + 1)/2)) \tag{31}$$

As such, the growth rate of average income of the bottom 40% is defined as the average income gross growth rate $1 + g_{\bar{y},t+1}^{pc}$times the gross growth rate of the income share of the bottom 40% ($SB40_{t+1}/SB40_t$)

$$\begin{aligned} 1 + g_{\bar{40},t+1} &\equiv \frac{E(y_{t+1}^{pc}|y_{t+1}^{pc} < k_{t+1})}{E(y_t^{pc}|y_t^{pc} < k_t)} \\ &= (1 + g_{\bar{y},t+1}^{pc}) \frac{\Phi(\Phi^{-1}(0.4) - \sigma_{t+1})}{\Phi(\Phi^{-1}(0.4) - \sigma_t)} \\ &= (1 + g_{\bar{y},t+1}^{pc}) \frac{SB40_{t+1}}{SB40_t} \end{aligned} \tag{32}$$

where $g_{\bar{y},t+1}^{pc}$ is the economy-wide per capita growth rate as defined in Equation 23 and σ is the SD of the underlying normal distribution (which is a one-to-one transformation of the Gini coefficient by Equation 20).

The *Shared Prosperity Premium (SPP)* (Equation 33) is the excess income growth of the bottom 40% ($g_{\bar{40},t+1}$) over the average per capita growth rate of the whole economy ($g_{\bar{y},t+1}^{pc}$):

$$SPP_{t+1} \equiv ln(1 + g_{\bar{40},t+1}) - ln(1 + g_{y,t+1}^{pc}) \approx g_{\bar{40},t+1} - g_{\bar{y},t+1}^{pc} \tag{33}$$

Combining Equations 33 and 32, one can see gain an expression for the SPP_{t+1}, which is just the log growth rate of income share of the bottom 40%.

$$SPP_{t+1} = ln\left[\frac{\Phi(\Phi^{-1}(0.4) - \sigma_{t+1})}{\Phi(\Phi^{-1}(0.4) - \sigma_t)}\right] \tag{34}$$

$$= ln\left[\frac{SB40_{t+1}}{SB40_t}\right] \tag{35}$$

From Equation 34 one can see that when there is no change in income inequality (such that $\sigma_{t+1} = \sigma_t$ and $G_{t+1} = G_t$), then the shared prosperity premium will be zero, and the growth rate of the incomes of the bottom 40% will be the same as the per capita growth rate of the economy as a whole (recall from equation 22 that $\sigma_t = \sqrt{2} \times \Phi^{-1}([G_t + 1]/2)$). As such, if income follows a log-normal distribution then there is almost a one-to-one relationship between the change in inequality (as measured by the Gini coefficient) and the shared prosperity premium: a fall (raise) in inequality is equivalent to a positive (negative) shared prosperity premium.

[18]This follows from the expression for a conditional mean of a log normal distribution: $E(X|X < k_t) = e^{\mu+\sigma^2/2}\Phi(\frac{ln(k_t)-\mu_t-\sigma_t^2}{\sigma_t})/\Phi(\frac{ln(k_t)-\mu_t}{\sigma_t})$, where $\Phi(\frac{ln(k_t)-\mu_t}{\sigma_t}) = 0.4$

[19]To see this, we normalize the population to 1, which means that the mean per capita income $\bar{y}^{pc}$ equals total income.

Shared Prosperity Implementation in the LTGM Spreadsheet

Given Equation 34, the Shared Prosperity Premium enters the LTGM as an alternative way for the user to specify the path of inequality, or to summarize the implications for the bottom 40% of a given path of Gini coefficient. As such, the shared prosperity mostly enters in the sheet InputdataA when the user is specifying the path for inequality (and the SPP is plotted in GraphsA).[20]

- If the user specifies a path for the Gini coefficient, then the implied Shared Prosperity Premium is calculated using Equation 34 as a residual (using Equation 22 to substitute out for the Gini as an intermediate step).
- If the user specifies a path for the Shared Prosperity Premium (SPP):
  - the user must still specify an initial Gini coefficient G_t for the first year, which is then converted into an initial σ_t using Equation 22.
  - σ_{t+1} given by Equation 36, where a higher SPP increases reduces σ_{t+1}

$$\sigma_{t+1} = \Phi^{-1}(0.4) - \Phi^{-1}\left[e^{SPP_{t+1}}\Phi(\Phi^{-1}(0.4) - \sigma_t)\right] \tag{36}$$

  - the Gini coefficient G_{t+1} (which enters the model spreadsheets) is calculated using Equation 20.
- The income share of bottom 40% of the population ($SB40$) is recorded as a memorandum item using Equation 30.

References

[1] Bourguignon, F., 2007, "The Growth Elasticity of Poverty Reduction: Explaining Heterogeneity across Countries and Time Periods" in Eicher and Turnovsky (eds.) *Inequality and Growth: Theory and Policy Implications* MIT Press

[2] Hevia C. and N. Loayza, 2012, "Savings and Growth in Egypt" *Middle East Development Journal* 4, 1

[3] Inklaar R. and M. Timmer, 2013, "Capital, labor and TFP in PWT8.0" (July 2013) [link]

[4] Lopez H. and L. Serven, 2006, "A Normal Relationship? Poverty, Growth, and Inequality" World Bank Policy Research Working Paper 3814, January 2006

[5] Solow, R. 1956, "A Contribution to the Theory of Growth", *Quarterly Journal of Economics*, 70:65-94.

[6] Swan, T. 1956, "Economic Growth and Capital Accumulation", *Economic Record*, 32:334-361.

[20] The income growth of the bottom 40% in each of Model 1/1s/2/2s/3/3s is also summarized at a poverty memorandum item at the bottom of each of those sheets and plotted in GraphsB.