Folder Title: Correspondence of Economic Adviser Hollis B. Chenery - Correspondence 02

Folder ID: 1784955

ISAD(G) Reference Code: WB IBRD/IDA 03 EXC-10-4483S

Series: Correspondence of Economic Advisor Hollis Chenery

Sub-Fonds: Records of President Robert S. McNamara

Fonds: Records of the Office of the President

Digitized: March 19, 2013

Chenery Files - Development Research Center, Sep. 1971 - Jan. 1972

May 25, 1972

MR. CHENERY:

Further checking in this office shows that this letter was sent to MR. GOREUX - original attached hereto. A copy of the further letter from MR. FERNANDEZ has been given to MR. GOREUX.

Ask Coreux to answer letter - Phoned through request p.r. 5/25/72

January, 1972

Preliminary Draft
Comments Welcomed
Not to be Quoted

ESTIMATION OF LLUCH's EXTENDED LINEAR EXPENDITURE SYSTEM FROM CROSS-SECTIONAL DATA

by

Alan Powell*

Contents

* The author is an economist at the Development Research Center, World Bank. He is very grateful to Constantino Lluch for his patient consultations and stimulus on the topic. The errors are the author's alone.

ESTIMATION OF LLUCH's EXTENDED LINEAR EXPENDITURE SYSTEM
FROM CROSS-SECTIONAL DATA

Abstract

Complete sets of demand relations may be fitted using varying types of sample information and varying a priori specifications. In this paper the identification and estimation of Lluch's extended linear expenditure system (ELES) from cross-sectional data alone is investigated. Under the most favorable conditions of data availability, all of the parameters of the ELES model are identified, and are estimable by the method of reduced form least squares. This is the case where observations on permanent income are available for the consuming units of the cross section and where, in addition, prices are recorded (even though they do not vary from one consuming unit to the next). Under the least favorable conditions only the marginal budget shares are identified. This corresponds to the case where no data on permanent income, or on savings, are available. The conventional ordinary least squares estimators of the marginal budget shares are, under these conditions, biased and inconsistent. Expressions are developed for the large-sample biases.

Acknowledgement

Many of the ideas presented here are to be found in Belandria ([1] , Ch. 4). The purpose of the present note is to put these results on a systematic basis and to integrate them with some new results on the large sample biases inherent in common specifications (such as [4] , [6] , [8]) of consumer expenditure equations.

The ELES Specification

In reference [5] Lluch shows that the intertemporal maximization of a Stone-Geary utility function [3] leads (under his behavioral specification) to the following set of commodity expenditure equations:

$$v_{it} = p_{it}\gamma_i + \mu\beta_i(z_t - \underline{p}_t'\underline{\gamma}) \qquad (i=1,\ldots, n) , \tag{1}$$

in which v_{it} is the expenditure of a representative consumer at time t (or of the t^{th} consumer at a given time, on the i^{th} item of consumption expenditure $(i=1,\ldots, n)$; p_{it} is the price of the i^{th} item at time t (or confronting the t^{th} consumer at a given point of time); z_t is an appropriately defined 'permanent income' variable comprising the total income of the representative consumer at time t (or of the t^{th} consumer at a given time) to which is added the present value of expected future changes in labor income as perceived by the representative consumer at time t (or by the t^{th} consumer at a given point of time); that is

[Definition of 'Permanent Income']

$$z_t \overset{def}{=} y_t + \rho w_t + L(\dot{\underline{y}}_t) , \tag{2}$$

where y_t is labor income, w_t is wealth (excluding human capital), ρ is the rate of return on wealth, $\dot{\underline{y}}_t$ is the sequence of expected changes in y_t into the indefinite future as perceived from viewpoint t, and L is a present value operator which capitalizes the expectational series $\dot{\underline{y}}_t$ at an appropriate rate of discount. The other symbols in (1) are interpreted as follows. The vector $\underline{p}_t$ is an n dimensional column showing the prices of all n consumption items relevant to t -- prime denotes transposition. The parameters of (1) are the γ_i of the Stone-Geary utility function (sometimes interpreted as subsistence minima quantum indexes of the various consumption items -- see, e.g., [8]). Collected as a column vector these n coefficients are written $\underline{\gamma}$. The parameter μ may be interpreted as the marginal propensity to consume out of permanent income z_t . The coefficients β_i are the marginal budget shares of the n consumption items in total consumption expenditure.

This extension of Stone's linear expenditure system achieves the integration of the complete systems approach to consumer demand with a model (albeit a simple one) of accumulation. The permanent income stream is treated exogenously; savings, however, are determined endogenously from the consumption function. The latter is obtained by simply summing (1) over commodities:

[Consumption Function]

$$v_t = (1-\mu)\underline{p}_t'\underline{\gamma} - \mu\, z_t \,, \tag{3}$$

in which v_t is total consumption expenditure, and where the identity forcing the marginal budget shares β_i to sum identically to unity has been used.

The specification of an error structure in more conventional versions of linear expenditure systems has always been complicated by the operation of the budget identity,

[Budget Identity]

$$v_t \equiv \underline{i}'\, \underline{v}_t \,, \tag{4}$$

in which $\underline{i}$ is an n-component column vector of units (the "summation vector"), and $\underline{v}_t$ is the column vector containing expenditures on the n items as appropriate to t. In the more conventional case (e.g. references [4], [6] and [8]), the variable v_t is predetermined and appears on the right of the expenditure equations analogous to (1). The incorporation of a series of stochastic disturbances into those equations requires degeneracy in their joint distribution in order to preserve the budget identity.

In the case of the present specification, it seems appropriate to introduce the disturbances into the systems equations (1), obtaining

[<u>Reduced Form of Commodity Expenditure System</u>]

$$v_{it} = p_{it}\,\gamma_i + \mu\beta_i(z_t - \underline{p}_t'\underline{\gamma}) + \varepsilon_{it} \qquad (i=1,\ldots,n). \tag{5}$$

Since permanent rather than actual income is involved, there is no iron law of aggregation acting across these commodity expenditure equations. Further, the exogeneity to this model of the series $\{z_t\}$ guarantees the plausibility of the assumption that z_t is independent of error equation disturbance. For future use we assume, therefore, that

$$\text{Cov}\,(z_t\,,\,\varepsilon_{it}) = 0 = \text{plim}\left(\frac{1}{T}\sum_{t=1}^{T} z_t\,\varepsilon_{it}\right) \quad (\text{for all } i)\;. \tag{6}$$

Consider the case in which the sample is a time-series. If the ε_{it} are free from within-equation and between-equations serial correlation, then the covariance structure is fully characterized by the contemporaneous variance-covariance matrix of the ε's; that is, by

$$E\,\underline{\varepsilon}_t\,\underline{\varepsilon}_t' \overset{\text{def}}{=} \underset{n\times n}{\underline{\underline{\Sigma}}} \qquad (\text{for all } t), \tag{7}$$

where $\underline{\varepsilon}_t$ is the n-component vector of stochastic disturbances in the commodity expenditure equations at data point t. In the case of cross-sectional data, the matrix $\underline{\underline{\Sigma}}$ surely gives an adequate representation of the error structure. As indicated above, there is no reason to suppose that any restrictions operate across the rows or columns of $\underline{\underline{\Sigma}}$; that is, we assume

$$\text{rank } (\underline{\underline{\Sigma}}) = n \quad . \tag{8}$$

Data on Permanent Income Available

Assume for the moment that data on z_t are available for a cross-sectional sample in which prices are unknown, but it is known that every consuming unit pays the same price. Then the i^{th} commodity expenditure equation (5) may be rewritten

$$v_{it} = \theta_i + \Phi_i z_t + \varepsilon_{it} \qquad (i=1,\ldots, n) \tag{9.1}$$

in which

$$\theta_i = p_i \gamma_i - \mu\beta_i \underline{p}'\underline{\gamma} \qquad (p_{it} = p_i \text{ for all } t, \text{ by assumption}), \tag{9.2}$$

and

$$\Phi_i = \mu\beta_i \quad . \tag{9.3}$$

Equation (9.1) is the "identical regressors" problem in which every left-hand variable is regressed on the same set of exogenous variables $[\underline{i}\ ,\ \underline{z}]$. To put it slightly differently, (9.1) is one of n reduced form equations from the same structural system. Under the assumption that the ε_{it} are joint normally distributed, the full information maximum likelihood estimates of (9.1), therefore, can be obtained by the use of ordinary least squares applied on a commodity-by-commodity basis [2].

The consumption function obtained by summing the commodity expenditure equations (5) is

$$v_t = (1 - \mu)\underline{p}_t'\underline{\gamma} + \mu z_t + \underline{i}'\,\underline{\varepsilon}_t \quad . \tag{10}$$

The variance of the disturbance in this equation is, from (7),

$$\mathrm{Var}(\underline{i}'\underline{\varepsilon}_t) = \underline{i}'\,\underline{\underline{\Sigma}}\,\underline{i} \quad . \tag{11}$$

The maximum likelihood estimate of μ is obtainable by adding the estimated Φ_i 's:

$$\hat{\mu} = \sum_{i=1}^{n} \hat{\Phi}_i = \underline{i}'\hat{\underline{\Phi}} \qquad \text{because} \left(\sum_{i=1}^{n} \beta_i \equiv 1\right) . \tag{12}$$

The maximum likelihood estimates of the marginal budget shares β_i are then obtainable as

$$\hat{\beta}_i = \hat{\Phi}_i / (\underline{i}'\hat{\underline{\Phi}}) \quad . \tag{13}$$

In the cross-sectional context, adding the estimated θ's across equations gives

$$\text{MLE of } \left\{(1 - \mu)\underline{p}'\underline{\gamma}\right\} = \underline{i}'\hat{\underline{\theta}} \quad . \tag{14}$$

The maximum likelihood estimate of $(\underline{p}'\underline{\gamma})$ is obtained as

$$\text{MLE of } (\underline{p}'\underline{\gamma}) = \underline{i}'\hat{\underline{\theta}} / (1 - \underline{i}'\hat{\underline{\Phi}}) \quad . \tag{15}$$

Finally, estimates of $(p_i\gamma_i)$ are obtained from expression (9.2) as

$$\text{MLE of } (p_i\gamma_i) = \hat{\theta}_i + \hat{\Phi}_i\,\underline{i}'\hat{\underline{\theta}} / (1 - \underline{i}'\hat{\underline{\Phi}}) \ . \tag{16}$$

Notice that it is not necessary to have price variation over the subscript t in order to identify the γ_i 's; it is necessary to have measurements on the p_i (that is, of the prices of the n different consumption items).

For the above method to be operational, cross-sectional data on permanent income are needed. One approach is that of Belandria [1] who uses socio-economic variables available in his cross-sectional sample data in order to develop proxy data for z_t .

Data on Savings Available

Define actual income as

$$x_t \overset{\text{def}}{=} y_t + \rho w_t \quad , \tag{17}$$

the sum of labor and non-labor income. In some cross-sectional surveys, data on x_t are collected. In some situations it may be reasonable to suppose that the present value of expected future gains in non-labor income bears a constant ratio to actual income for all consumption units surveyed; that is, it may be reasonable to assume

$$L(\dot{\underline{y}}_t) = a\, x_t \ , \tag{18}$$

where a is not a function of t. Then, from (2) and (17),

$$z_t = (1 + a)x_t \quad . \tag{19}$$

The consumption function, (10), becomes

$$v_t = (1-\mu)\underline{p}_t'\underline{\gamma} + \mu(1+a)x_t + \underline{i}'\underline{\varepsilon}_t \quad , \tag{20}$$

whilst the commodity expenditure equations (9.1) become

$$v_{it} = \theta_i + (1+a)\Phi_i\, x_t + \varepsilon_{it} \tag{21}$$

$$= \theta_i + \eta_i x_t + \varepsilon_{it} \qquad (i=1,\ldots,n)$$

(say). Then whilst the β_i are identified, and may be estimated as

$$\hat{\beta}_i = \hat{\eta}_i / (\underline{i}'\hat{\underline{\eta}}) , \qquad \left[\hat{\underline{\eta}} \overset{\text{def}}{=} (\hat{\eta}_1,\ldots, \hat{\eta}_n)\right] , \tag{22}$$

none of a, μ, or the set $\{\gamma_i\}$ are identifiable without further information (such as would be provided by price variation in the sample).

Only Expenditure Data Available

For many cross-sectional sets of data, only information on expenditures is collected. In this case one can no longer work with reduced form equations such as (9.1) and (10), but must work instead in equations which replace z_t on the RHS of (9.1) by total expenditure v_t (for which data are available). From (10) we see that

$$z_t = (v_t - (1-\mu)\underline{p}_t'\underline{\gamma} - \underline{i}'\underline{\varepsilon}_t) / \mu \quad . \tag{23}$$

Substituting from (23) into the RHS of (5),

$$v_{it} = (p_i\gamma_i - \beta_i\underline{p}_t'\underline{\gamma}) + \beta_i v_t + (\varepsilon_{it} - \beta_i\underline{i}'\underline{\varepsilon}_t) \quad . \tag{24}$$

Keeping in mind that $\underline{p}_t$ does not vary across t for cross-sectional data, (24) may be written

$$v_{it} = \alpha_i + \beta_i v_t + e_{it} \quad . \tag{25}$$

This is the form in which linear expenditure functions are commonly fitted to data lacking price variation. The budget identity (4) is sufficient to ensure that the ordinary least squares estimators of the α's and β's satisfy[1/]

$$\sum_{i=1}^{n} \hat{\alpha}_i \equiv 0 \; ; \qquad \sum_{i=1}^{n} \hat{\beta}_i \equiv 1 \quad . \tag{26}$$

This is fortunate since the theory requires both of these results; the error structure of the system (25) is far from classical, however and the regressor v_t is endogenous. Summation of e_{it} over commodity equations is instructive:

$$\sum_{i=1}^{n} e_{it} \equiv \sum_{i=1}^{n} \varepsilon_{it} - \underline{i}'\underline{\varepsilon}_t \sum_{i=1}^{n} \beta_i \equiv 0 \quad . \tag{27}$$

Thus it is seen that the development of the linear expenditure equations (25) from (5) provides an economic and a statistical rationale for the degeneracy in the joint distribution of e_{it} (whereas in the standard treatment of (5) this degeneracy comes about from accounting necessity only).

Because (26) is guaranteed by ordinary least squares, the sums across equations of biases in the estimates of the α's and the β's vanish identically. This is of little comfort, however, since its operational implication is that a large positive bias in the estimate of the marginal budget share of one item must be offset by collectively large negative biases for other items. The large sample biases of the ordinary least squares estimates of the marginal budget shares can be analyzed along conventional lines as follows:

$$\hat{\beta}_i = \frac{1}{T} \sum_{t=1}^{T} v_{it}(v_t - \bar{v}) \Big/ \left[\frac{1}{T} \sum_{t=1}^{T} v_t(v_t - \bar{v})\right] , \tag{28}$$

1/ See, e.g., [7] .

in which T is the sample size and $\bar{v}$ is the sample mean of total expenditure. Substituting from (25) into (28) we see that

$$\text{Numerator of (28)} = \alpha_i \sum_{t=1}^{T} (v_t - \bar{v})/T + \beta_i \sum_{t=1}^{T} v_t(v_t - \bar{v})/T + \sum_{t=1}^{T} e_{it}(v_t - \bar{v})/T \tag{29}$$

The first term on the RHS of (29) vanishes identically, whilst

$$\text{plim}\left[\sum_{t=1}^{T} v_t(v_t - \bar{v})/T\right] = \text{Var}(v_t) \quad , \tag{30.1}$$

and

$$\text{plim}\left[\sum_{t=1}^{T} e_{it}(v_t - \bar{v})/T\right] = \text{Covar}(v_t , e_{it}) \quad . \tag{30.2}$$

Consequently

$$\text{plim}(\hat{\beta}_i) = \frac{\beta_i \text{Var}(v_t) + \text{Covar}(v_t , e_{it})}{\text{Var}(v_t)} \tag{31}$$

$$= \beta_i + \frac{\text{Covar}(v_t , e_{it})}{\text{Var}(v_t)} \quad .$$

Now, from (10) and (24),

$$\text{Covar}(v_t , e_{it}) = E\left[(\mu z_t + \underline{i}'\underline{\varepsilon}_t)(\varepsilon_{it} - \beta_i \underline{i}'\underline{\varepsilon}_t)\right] , \tag{32}$$

$$= E(\varepsilon_{it}\, \underline{i}'\underline{\varepsilon}_t) - \beta_i E\left[(\underline{i}'\underline{\varepsilon}_t)^2\right]$$

since the ε_{it} and z_t are, by assumption, independent. Rewrite the contemperaneous variance covariance matrix of the ε's as

$$\underline{\underline{\Sigma}} = \left[\underline{\sigma}_1 \mid \underline{\sigma}_2 \mid \dots \mid \underline{\sigma}_n\right] = \begin{bmatrix} \underline{\sigma}_1' \\ \vdots \\ \underline{\sigma}_n' \end{bmatrix} \quad . \tag{33}$$

Then (32) reduces to

$$\text{Covar}(v_t\ ,\ e_{it}) = \underline{i}'\underline{\sigma}_i - \beta_i\ \underline{i}'\ \underline{\underline{\Sigma}}\ \underline{i} \ . \tag{34}$$

From (10), the variance of v_t is

$$\text{Var}(v_t) = \mu^2\ \text{Var}(z_t) + \underline{i}'\ \underline{\underline{\Sigma}}\ \underline{i} \ . \tag{35}$$

For the purpose of this discussion, define the large sample bias (LSB) of $\hat{\beta}_i$ by

$$\text{LSB}\ (\hat{\beta}_i) = \text{plim}\ (\hat{\beta}_i) - \beta_i \ . \tag{36}$$

From (31), (34) and (35), the large sample bias is

$$\text{LSB}\ (\hat{\beta}_i) = (\underline{i}'\underline{\sigma}_i - \beta_i\ \underline{i}'\ \underline{\underline{\Sigma}}\ \underline{i}\ ) \,/\, (\mu^2\ \text{Var}(z_t) + \underline{i}'\ \underline{\underline{\Sigma}}\ \underline{i}) \ . \tag{37}$$

Because the β_i and $\hat{\beta}_i$ both add across i to unity, the sum of these biases should be zero. Checking, we see that since

$$\sum_{i=1}^{n} \underline{i}'\underline{\sigma}_i \equiv \underline{i}'\ \underline{\underline{\Sigma}}\ \underline{i} \ , \tag{38}$$

the requirement

$$\sum_{i=1}^{n} \text{LSB}(\hat{\beta}_i) \equiv 0 \tag{39}$$

does, in fact, hold.

If values can be found for these biases, then the ordinary least squares estimators can be corrected to yield estimators which are consistent. As the marginal propensity to consume, μ , is not identifiable from the type of data we are discussing, an extraneous estimate would be needed, as would also be the case with the variance of permanent income $Var(z_t)$. From the nature of the problem, only inconsistent estimates of the β_i and $\underline{\underline{\Sigma}}$ would be available initially. Hence only rather rough (and inconsistent) estimates of the LSB's would be available -- the correction, however, may still be worth making.

REFERENCES

[1] Belandria, Francisco, "An Empirical Study of Consumer Expenditure Patterns in Venezuelan Cities", unpublished Ph.D. dissertation, Northwestern University (August 1971), pp. viii and 146 (mimeo).

[2] Dhrymes, Phoebus, Econometrics -- Statistical Foundations and Applications (New York: Harper and Row, 1970), pp. 153-161.

[3] Geary, R. C., "A Note on 'A Constant-Utility Index of the Cost of Living", Review of Economic Studies, Vol. 18 (1949-50), pp. 65-66.

[4] Leser, C. E. V., "Demand Functions for Nine Commodity Groups in Australia", Australian Journal of Statistics, Vol. 2, No. 3 (November 1960), pp. 102-113.

[5] Lluch, Constantino, "The Extended Linear Expenditure System", University of Essex, Department of Economics Discussion Paper, No. 6 (February 1970), pp. 8.

[6] Powell, Alan, "A Complete System of Demand Equations for the Australian Economy Fitted by a Model of Additive Preferences", Econometrica, Vol. 34, No. 3 (July 1966), pp. 661-675.

[7] Powell, Alan, "Aitken Estimators as a Tool in Allocating Predetermined Aggregates", Journal of the American Statistical Association, Vol. 64, No. 327 (September 1969), pp. 913-922.

[8] Stone, Richard, "Linear Expenditure Systems and Demand Analysis: An Application to the British Pattern of Demand", Economic Journal, Vol. 64, No. 255 (September 1954), pp. 511-27.

20 January 1972

Partial First Draft/Confidential
Comments Welcomed

ESTIMATION OF LLUCH's EXTENDED LINEAR EXPENDITURE SYSTEM FROM TIME-SERIES DATA

by

Alan Powell*

* The author is an economist at the Development Research Center, World Bank. He is very grateful to Constantino Lluch for his patient consultations and stimulus on the topic. The World Bank is not responsible for any views expressed in this paper. The errors are the author's alone.

Introduction

The purpose of this paper is to specify an operational procedure for estimating Lluch's expanded linear expenditure system (ELES; [ ], [ ]) from aggregate time-series data.

In Section 1 the estimation of the consumption function is examined. For convenience of exposition, this discussion proceeds as if a sequence of observations on the appropriate permanent income variable were available. Given the availability of such a series, as well as time series on price indexes for the n consumption items of the system, many systems parameters are identifiable and efficiently estimable from the fitted consumption function. (The exceptions are the marginal budget shares.)

In Section 2 the question of constructing a permanent income series is taken up. The maximum likelihood determination of such a series conjointly with the estimation of the consumption function parameters is outlined.

1. Basic Principles for Estimation of the Consumption Function

The ELES Specification

The commodity expenditure system under Lluch's ELES specification is as follows:

$$v_{it} = p_{it}\,\gamma_i + \mu\beta_i(z_t - \underline{p}_t'\underline{\gamma}) + \varepsilon_{it} \qquad (i=1, \ldots, n) \qquad (1)$$

in which v_{it} is the expenditure of a typical consumer at time t on the i^{th} of n possible consumption items; p_{it} is the price at t of the i^{th} such item, while $\underline{p}_t$ collects into a single column vector the prices at t of all n consumer goods; γ_i is the parameter of the Stone-Geary utility function specific to the i^{th} consumption item, whilst $\underline{\gamma}$ is the column vector containing all n such parameters; μ is the aggregate marginal propensity to consume out of appropriately defined permanent income, z_t ; β_i is the marginal budget share of consumption item i ; and ε_{it} is a zero-mean stochastic element which is assumed independent of each p_{jt} $(j=1, \ldots, n)$ and of z_t . The permanent income variable z_t is defined as the sum of current income (from all sources) plus the present value of expected future gains in labor income; i.e.,

$$z_t = x_t + L(\dot{y}_t) \qquad (2)$$

where x_t is current income (from all sources); $\dot{y}_t$ is a series of the changes in labor income (wages and salaries) expected from the viewpoint of t to prevail at $(t+1)$, $(t+2)$, ...; and L is the present value operator which discounts income streams at the rate ρ .

If one sums the expenditure equations (1) over consumption items, (keeping in mind that the β_i sum to unity) one obtains the following consumption function:

$$\underset{T\times1}{\underline{v}} = (1 - \mu)\underset{T\times n}{\underline{\underline{P}}}\,\underset{n\times1}{\underline{\gamma}} + \mu\underset{T\times1}{\underline{x}} + \mu L(\underset{T\times1}{\dot{\underline{y}}}) + \underset{T\times n}{\underline{\underline{E}}}\,\underset{n\times1}{\underline{i}} , \tag{3}$$

where $\underline{v}$ = total consumption expenditures (a Tx1 vector with typical element $v_t = \sum_i v_{it}$);

$\underline{\underline{P}}$ = a matrix whose $(t, j)^{th}$ element p_{tj} is the price of item j at time t ;

$\underline{x}$ = a vector of T sample observations x_t on total income at $t=1, \ldots, T$;

$\dot{\underline{y}}$ = a (Tx1) vector whose t^{th} element is <u>a series</u> of the changes in labor income expected to prevail from viewpoints $t=1, \ldots, T$ at future points of time $(t+1), (t+2), \ldots$;

$\underline{\underline{E}}$ = a matrix whose $(t, j)^{th}$ element is the stochastic error term ε_{tj} (appearing at t^{th} observation of equation for j^{th} expenditure item);

and $\underline{i}$ = $(1, 1, \ldots, 1)'$, the n-dimensional summation vector.

(As previously defined, μ is the marginal propensity to consume.)

<u>Stochastic Specification</u>

Hopefully, the ε_{it} of equation (1) will be free of serial correlation (both within and between expenditure equations). If this is the case, then the covariance structure of the system may be characterized as follows. If by $\underline{E}$ is defined the $(n\ T)$ by 1 super-vector whose elements are identically those of $\underline{\underline{E}}$ after rearrangement as

$$\underline{E} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{T1} \\ \hline \varepsilon_{12} \\ \varepsilon_{22} \\ \\ \varepsilon_{T2} \\ \hline \vdots \\ \hline \varepsilon_{1n} \\ \varepsilon_{2n} \\ \vdots \\ \varepsilon_{Tn} \end{bmatrix} , \qquad (4)$$

then

$$E(\underline{E}\,\underline{E}') = \underset{n\times n}{\underline{\underline{\Sigma}}} \otimes \underset{T\times T}{\underline{\underline{I}}} , \qquad (5)$$

in which E is the expectations operator; $\otimes$ is the Kronecker product operator; and $\underline{\underline{\Sigma}}$ is the contemporaneous variance-covariance matrix of the system, defined as follows:

$$\underline{\underline{\Sigma}} = [\sigma_{ij}] ; \qquad (6.1)$$

$$\sigma_{ij} = E(\varepsilon_{it}\,\varepsilon_{jt}) . \qquad (6.2)$$

Let

$$\underline{\varepsilon}_t' = (t^{th} \text{ row of } \underline{\underline{E}}) \qquad (7)$$

Then

$$(t^{th} \text{ element of } \underline{\underline{E}}\,\underline{i}) = \underline{\varepsilon}_t'\underline{i} = \underline{i}'\underline{\varepsilon}_t = e_t \text{ (say)}. \qquad (8)$$

Let the Tx1 sample vector on e_t be denoted $\underline{e}$. Then by virtue of assumption (5),

$$\underset{T\times T}{E\, \underline{e}\, \underline{e}'} = (\underline{i}' \, \underline{\underline{\Sigma}} \, \underline{i}) \, \underline{\underline{I}}_T \quad . \tag{9}$$

Estimation of Consumption Function

If in (1) the errors ε_{it} are joint normally distributed, then e_t is normal also and the maximum likelihood estimators[1] (MLE's) of $\{(1-\mu)\underline{\gamma} \; ; \mu\}$ are obtained by regressing total expenditure v_t on the set of n prices and on permanent income z_t :

$$\begin{aligned} \underline{v} &= \underline{\underline{P}}\, \gamma(1-\mu) + \mu \underline{z} + \underline{e} \\ &= \underline{\underline{X}}\, \underline{\tau} + \underline{e} \quad \text{(say)}, \end{aligned} \tag{10}$$

where

$$\underset{T\times(n+1)}{\underline{\underline{X}}} = [\underset{T\times n}{\underline{\underline{P}}} \mid \underset{T\times 1}{\underline{z}}] \quad , \tag{11.1}$$

and

$$\underline{\tau} = \begin{bmatrix} \underline{\gamma}(1-\mu) \\ \mu \end{bmatrix} \quad . \tag{11.2}$$

If $\hat{\tau}_j$ is the j^{th} ordinary least-squares (OLS) estimate contained in the OLS estimating vector $\hat{\underline{\tau}}\,[=(\underline{\underline{X}}'\underline{\underline{X}})^{-1}\underline{\underline{X}}'\underline{v}]$, then the MLE of γ_j is obtained as

$$\hat{\gamma}_j = \hat{\tau}_j / \hat{\tau}_{n+1} \qquad (j=1, \ldots, n) \; . \tag{12}$$

1/ Strictly speaking, these are the conditional maximum likelihood estimators given the data on prices and permanent income. Throughout the discussion of Section 1, this interpretation should be borne in mind.

For notational simplicity, write

$$\sigma^2 \equiv \underline{i}' \, \underline{\underline{\Sigma}} \, \underline{i} \; . \tag{13}$$

The MLE of σ^2 (that is, of the variance of e) is simply the mean of the squared residuals from the OLS fit:

$$\begin{aligned} \sigma^2 &= T^{-1} \, \hat{\underline{e}}' \hat{\underline{e}} \\ &= T^{-1} (\underline{v}' \underline{v} - \hat{\underline{T}} \, \underline{\underline{X}}' \, \underline{v}) \; . \end{aligned} \tag{14}$$

This estimator is not correlated with any element of $\hat{\underline{T}}$ -- to see this, it is only necessary to examine the expected product of the sampling error of $\hat{\underline{T}}$ with $\hat{\sigma}^2$. From Goldberger [, p. 164],

$$\hat{\underline{T}} - E(\hat{\underline{T}}) = (\underline{\underline{X}}' \underline{\underline{X}})^{-1} \, \underline{\underline{X}}' \underline{e} \quad , \tag{15}$$

so that, from (15)

$$\begin{aligned} &\underset{(n+1)x1}{\underline{\text{Cov}}} \; (\hat{\sigma}^2 \, , \, \hat{\underline{T}}) \\ &\qquad = E(T^{-1} \hat{\underline{e}}' \hat{\underline{e}} \, . \, (\underline{\underline{X}}' \underline{\underline{X}})^{-1} \underline{\underline{X}}' \underline{e}) \; . \end{aligned} \tag{16}$$

But the OLS residuals $\hat{\underline{e}}$ are linear in the true errors (Goldberger [, p. 166]):

$$\hat{\underline{e}} = [\underline{\underline{I}} - \underline{\underline{X}} (\underline{\underline{X}}' \underline{\underline{X}})^{-1} \, \underline{\underline{X}}'] \underline{e} \quad , \tag{17}$$

in which (under our assumptions) the matrix $\underline{\underline{X}}$ is independent of $\underline{e}$. Because $\underline{e}$ is normal, the terms of (16) involving the cubes e_1^3 , ..., e_T^3 vanish (whilst the remaining terms vanish due to the serial independence of the e's).

Define

$$\underline{\hat{b}} = (\hat{f}_1, \ldots, \hat{f}_{n+1};\ \hat{\sigma}^2)' \ , \tag{18}$$

$$= \text{estimate of } [(1-\mu)\gamma_1, \ldots, (1-\mu)\gamma_n;\ \mu\ ;\ \sigma^2]' \ .$$

Then $\underline{\hat{b}}$ has variance-covariance matrix

$$\underset{(n+2)\times(n+2)}{\underline{\underline{\text{Var}}}\ (\underline{\hat{b}})} = \left[\begin{array}{c|c} \sigma^2(\underline{\underline{X}}'\underline{\underline{X}})^{-1} & 0 \\ \hline 0 & \text{Var}(\hat{\sigma}^2) \end{array}\right] \tag{19.1}$$

$$= \underline{\underline{B}} \quad \text{(say)} \ . \tag{19.2}$$

The (n+2) dimensional parameter vector fully characterizing the consumption function (10) is

$$\underline{\theta} = (\gamma_1\ ,\ \gamma_2\ ,\ \ldots,\ \gamma_n\ ;\ \mu\ ;\ \hat{\sigma}^2) \ . \tag{20}$$

The transformation linking $\underline{b}$ to $\underline{\theta}$ is

$$\theta = \left[\begin{array}{c|c|c} (1-\mu)^{-1}\ \underline{\underline{I}}_n & \underline{0} & \underline{0} \\ \hline \underline{0}' & 1 & \underline{0} \\ \hline \underline{0}' & \underline{0}' & 1 \end{array}\right] \underline{b} \quad ; \tag{21}$$

$$= \underline{\underline{A}}\ \underline{b} \qquad \text{(say)} \ .$$

Note that some elements of $\underline{\underline{A}}$ are functions of an element of $\underline{b}$. For future use we note that the Jacobian of the transformation (21) is

$$\underset{(n+2)\times(n+2)}{\partial\underline{\theta}\ /\ \partial\ \underline{b}} = [\partial\theta_i / \partial b_j] \tag{22}$$

$$= \underline{\underline{A}} + [\underline{0},\ \underline{0},\ \ldots,\ \underline{0}\ ;\ \underline{\underline{C}}\ \underline{b};\ \underline{0}] \ ,$$

in which

$$\underline{\underline{C}} = \left[\begin{array}{c|c|c} (1-\mu)^{-2}\,\underline{\underline{I}}_n & \underline{0} & \underline{0} \\ \hline \underline{0}' & 0 & 0 \\ \hline \underline{0}' & 0 & 0 \end{array}\right] \quad . \tag{23}$$

The MLE of $\underline{\theta}$ is obtained by applying (12) to the OLSE's in order to obtain the first n elements, and by using (14) to obtain the last element. (The MLE of the $(n+1)^{th}$ element -- namely, of μ -- is just the OLSE itself.) Denote this ML estimating vector for $\underline{\theta}$ by $\hat{\underline{\theta}}$. The asymptotic variance-covariance matrix for $\hat{\underline{\theta}}$ is (Goldberger, [, p. 125]),

$$\underset{(n+2)\times(n+2)}{\underline{\underline{\text{Var}}}\ (\hat{\underline{\theta}})} = (\partial\underline{\theta} / \partial\underline{b})'\underline{\underline{B}}(\partial\theta / \partial\underline{b}) \ . \tag{24}$$

This matrix may be estimated consistently by replacing the unknown elements of (24) by their MLE's (Dhrymes [, p. 136]).

2. Expectational Framework and Permanent Income Series

Before proceeding to the operational specification of permanent income z_t , some further notation is necessary. Actual labor income, a time series, will be denoted $y(t)$. This is to be distinguished sharply from the values anticipated for the future at given points of time. The labor income expected to pertain in some future period τ from the viewpoint of a consumer at time t will be written $y_t(\tau)$. When particular expectations hypotheses are being explored -- that is, when special cases are being proposed for an operational version of the conjectural series $y_t(\tau)$ -- the resulting series will be denoted $\hat{y}_t(\tau)$, $\hat{\hat{y}}_t(\tau)$, $\tilde{y}_t(\tau)$; etc.

Linear Expectations Framework

The linear[1] expectations model for labor income expected, from the viewpoint of period t, to pertain in (t+1), is

$$y_t(t+1) = \sum_{j=0}^{\infty} W_{j1}\, y(t-j) \ , \tag{25}$$

where

$$\sum_{j=0}^{\infty} W_{j1} = 1 \ . \tag{26}$$

Note that, in (25), data on $y(t)$ are assumed to be available for the formation of expectations during t.

For operational purposes, the vector $\dot{\underline{y}}$ will be cast in terms of first differences rather than time derivatives. Consequently the gain in labor

---

1/ Throughout this discussion, "linear" means linear in the data (not necessarily in the parameters).

income expected from the viewpoint of t to accrue between τ and $(\tau+1)$ will be written

$$\dot{y}_t(\tau+1) = y_t(\tau+1) - y_t(\tau) . \tag{27}$$

For the particularization (25) this becomes

$$\begin{aligned}\dot{\hat{y}}_t(t+1) &= \hat{y}_t(t+1) - y(t) \\ &= \sum_{j=1}^{\infty} W_{j1}\, y(t-j)-(1-W_{01})y(t) .\end{aligned} \tag{28}$$

In the log-linear expectations framework,linear combinations of the logarithms of labor income are taken, and then exponentiated:

$$y_t(t+1) = \exp\left[\sum_{j=1}^{\infty} W^{o}_{j1} \; \ln y(t-j)\right] . \tag{29}$$

(As before, the weights sum over j to unity.) For series which tend to grow exponentially, (29) is much more plausible than (25).

Linear Adaptive Expectations

A subcase of the linear expectations hypothesis is the adaptive case under which the weights W_{j1} are assumed to be non-negative:

$$W_{j1} \geq 0 \qquad \text{(for all } j) . \tag{30}$$

The best known example of a linear adaptive framework is the use by Nerlove [ ] of the geometric lag distribution in which

$$W_{j1} = \alpha(1-\alpha)^{j-1} \qquad (0 < \beta < 1 \; ; \; j=0, 1, \ldots). \tag{31}$$

Consider expectations being formed for y in (t+1) from the viewpoint of the end of period t. According to the geometric version (31) of the adaptive expectations framework, the expected y for (t+1) from the viewpoint of t is

$$\tilde{y}_t(t+1) = \alpha \sum_{j=0}^{\infty} (1-\alpha)^j \, y(t-j) \; . \tag{32}$$

Apart from its non-linearity in its parameter α, this formulation has a serious drawback; namely, its limited horizon. There is no obvious way of generalizing (32) to generate expectations on y in periods t+2, t+3, ..., etc., from the viewpoint of period t. If the series $\{y_t\}$ has a stationary mean, one might suppose

$$\tilde{y}_t(t+1) = \tilde{y}_t(t+2) = \tilde{y}_t(t+3) = \ldots \tag{33}$$

-- however, in most economies labor income (hopefully) will be growing over time, and (33) is totally implausible.

Linear Extrapolative Expectations

To reiterate, in the consumption function (4), the t^{th} element of $L(\dot{\underline{y}})$ is the present value of expected gains in labor income in all future periods as perceived from the viewpoint of period t. Then if by L_t we denote the t^{th} element of $L(\dot{\underline{y}})$, we have

$$L_t = L[\dot{y}_t(\tau) \; ; \; \tau = t+1, \ldots, \infty] \tag{34}$$

where unsubscripted L, as before, is the present value operator. For an operational definition of $y_t(\tau)$ we reject the adaptive framework

(a) because $y_t(\tau)$ is not well-defined for $\tau > t+1$; and

(b) because, in the context of a growing series $\{y(t)\}$, the restriction of non-negative weights W rules out the possibility of even the most straightforward extrapolation of past growth experience into the future (Powell [ ], Ch. 6).

The sub-case of (25) and (26) in which

$$W_{01} > W_{11} > W_{21} > \cdots \tag{35}$$

and in which

$$W_{j1} < 0 \qquad \text{(for some } j) \tag{36}$$

will be termed "the linear extrapolative expectations model".

For operational purposes, the infinite limit of summation in (25) is an embarrassment. Without attempting at this point how to determine it, let us define t^* to be effective length of the memory process. [That is, historical experience of $(t^* + 1)$ or more periods ago is completely discounted insofar as the formation of expectations is concerned.] The operational version of (25) becomes

$$y_t(t+1) = \sum_{j=0}^{t^*-1} W_{j1}\, y(t-j) \,. \tag{37}$$

One method of carrying out this extrapolation would be to fit an ordinary least-squares trend line to the last t^* periods' data on labor income, and to project one period into the future along the fitted trend line. If the fitted intercept and trend slope are $\hat{a}_t$ and $\hat{b}_t$ respectively, then (37) becomes[1]

$$\hat{y}_t(t+1) = \hat{a}_t + (t+1)\hat{b}_t \quad . \tag{38}$$

To demonstrate that (38) is indeed a special case of (37), let us denote the regressor matrix for the trend calculation at t by $\underline{\underline{T}}_t^o$. Then

$$\underset{2\mathrm{x}t^*}{\underline{\underline{T}}_t^o} = \begin{bmatrix} 1 & t-t^*+1 \\ 1 & t-t^*+2 \\ \vdots & \vdots \\ 1 & t-1 \\ 1 & t \end{bmatrix} \quad . \tag{39.1}$$

The regressand vector is

$$\underset{t^*\mathrm{x}1}{\underline{y}_t^o} = \begin{pmatrix} y_{t-t^*+1} \\ \vdots \\ y_t \end{pmatrix} \quad , \tag{39.2}$$

with

$$(\underline{\underline{T}}_t^o)'(\underline{\underline{T}}_t^o) = \begin{bmatrix} t^* & \sum\limits_{j=t-t^*+1}^{t} j \\ \sum\limits_{j=t-t^*+1}^{t} j & \sum\limits_{j=t-t^*+1}^{t} j^2 \end{bmatrix} \tag{39.3}$$

as the sample moment matrix of the regressors.

1/ The use of the symbol b in this Section differs from its use in Section 1.

The summations on the right hand side (RHS) of (39.3) can be simplified to yield

$$(\underline{\underline{T}}_t^o)'\underline{\underline{T}}_t^o = \begin{bmatrix} t^* & t^*\left[t - \frac{1}{2}(t^*-1)\right] \\ t^*\left[t-\frac{1}{2}(t^*-1)\right] & \frac{t^*}{6}\left[(2t+1)(2t+1-t^*) + 2(t+1-t^*)(t-t^*)\right] \end{bmatrix} \tag{39.4}$$

The estimated trend parameters are obtained as solutions of the OLS equation; namely

$$\begin{pmatrix} \hat{a}_t \\ \hat{b}_t \end{pmatrix} = \left[\left(\underline{T}_t^o\right)'\underline{T}_t^o\right]^{-1}\left(\underline{T}_t^o\right)'\underline{y}_t^o \quad . \tag{40}$$

If this fitted OLS trend is used to project labor income into the future, the expectational model is

$$\hat{y}_t(\tau) = \hat{a}_t + \hat{b}_t\tau \qquad (\tau=t+1,\ t+2,\ \ldots) \ . \tag{41}$$

Define $\underline{j}$ to be the column vector containing the first t^* positive integers as elements; let $\underline{i}$ be a column vector of t^* units. Then the solution of (40), when substituted into (41), yields

$$y_t(\tau) = [\underline{W}_1' + (\tau - t)\underline{W}_2']\underline{y}_t^o \tag{42.1}$$

in which

$$\underline{W}_1 = [2(2t^* + 1)\underline{i} - 6j] \,/\, [t^*(t^* - 1)] \ , \tag{42.2}$$

and

$$\underline{W}_2 = 6\left[\frac{2}{(t^*+1)}\,\underline{j} - \underline{i}\right]\Big/\left[t^*(t^* - 1)\right] \ . \tag{42.3}$$

Note that in these expressions the vectors $\underline{W}_1$ and $\underline{W}_2$ are functions of t^* only. The expressions also reveal the role of the second (unit) subscript on the scalar W's of (25) and its subsequent development: the expectational weights need a second subscript to indicate how far, $(\tau - t)$, into the future is the period for which the expectation is supposed to apply. (The first subscript refers to the past -- W_{ji} gives the weight of observed labor income $(j+1)$ periods ago in the linear expression for the expectation of labor income i periods hence.)

Let $\underline{y}$ be the entire $(T \mathrm{x} 1)$ data vector on labor income. By Q_t denote the linear operator which truncates $\underline{y}$ (top and bottom) in such a way that the last remaining observation is $y(t)$, whilst the first remaining observation is $y(t-t^*+1)$. [That is, Q_t operating on $\underline{y}$ has $\overset{o}{\underline{y}}_t$ of (39.2) as its operational product.] Also, by $\underline{W}_{(\tau-t)}$ denote the $(t^* \mathrm{x} 1)$ vector defined by

$$\underline{W}_{(\tau-t)} = \underline{W}_1 + (\tau - t)\underline{W}_2 \quad . \tag{43}$$

Then under the linear extrapolative hypothesis

$$\hat{y}_t(\tau) = \underbrace{[\underline{W}_1' + (\tau-t)\underline{W}_2']}_{1 \mathrm{~x~} t^*} \underbrace{Q_t(\underline{y})}_{t^* \mathrm{x} 1} \quad . \tag{44}$$

Consider now the first differences of (44):

$$\hat{\hat{y}}_t(\tau) = \hat{y}_t(\tau + 1) - \hat{y}_t(\tau) \qquad (\text{for } \tau = t+1, \ldots) \; . \tag{45}$$

From (38),

$$\hat{\hat{y}}_t(\tau) = \hat{b}_t \qquad (\text{for all } \tau \geqslant t+1) \; . \tag{46}$$

Recall that ρ is the rate of discount to be used in forming the present value series $\{L_t\}$. From (34),

$$\hat{L}_t = \sum_{\tau'=1}^{\infty} (1+\rho)^{-\tau'} \hat{\dot{y}}_t(t+\tau') \tag{47.1}$$

[from (46)]

$$= \hat{b}_t \sum_{\tau'=1}^{\infty} (1+\rho)^{-\tau'}$$

$$= \hat{b}_t / \rho \ , \tag{47.2}$$

where, from (41) and (42.1),

$$\hat{b}_t = \underline{W}_2' \, \underline{y}_t^o = \underline{W}_2' \, Q_t(\underline{y}) \ . \tag{47.3}$$

The problem of determining t^* aside, the above procedure is made fully operational by substituting $\hat{b}_t$ into (3), the t^{th} data point of which is

$$v_t = (1-\mu) \sum_{j=1}^{n} p_{tj} \gamma_j + \mu x_t + (\tfrac{\mu}{\rho}) \hat{b}_t + \underline{i}' \underline{\varepsilon}_t \ . \tag{48}$$

In his derivation [ ] of (3), Lluch shows that μ is to be interpreted as the ratio of the rate of pure time preference discount δ to the rate of reproduction ρ of capital; that is,1/

$$\mu = \delta / \rho \ . \tag{49}$$

Consequently (48) may be parameterized as

1/ It is also shown in [ ] that ρ is the appropriate rate at which to compute present values L_t.

$$v_t = (1-\mu)\sum_{j=1}^{n} p_{tj}\,\gamma_j + \mu x_t + \Psi\hat{b}_t + \underline{i}'\underline{\varepsilon}_t \quad , \tag{50}$$

in which

$$\Psi = (\delta/\rho^2) \ . \tag{51}$$

The consumption function (50) may be estimated by ordinary least-squares yielding, as in Section 1, conditional MLE's. The MLE of ρ is

$$\hat{\rho} = \hat{\mu}/\hat{\Psi} \ , \tag{52}$$

where $\hat{\mu}$ and $\hat{\Psi}$ are the OLSE's from a fitted version of (50). The MLE of δ is

$$\hat{\delta} = (\hat{\mu})^2/\hat{\Psi} \ . \tag{53}$$

The sampling properties of $\hat{\rho}$ and $\hat{\delta}$ may be handled analogously with the development of equations (18) through (24) of Section 1.

To make this treatment complete, two further problems would need discussion; namely, (i) the determination of the effective memory length t^* ; and (ii) the accommodation of the idea of gradual (rather than abrupt) decay in the information value of historical experience for the formation of expectations. Both questions, however, are better left until the log-linear extrapolative model has been discussed.

<u>Log-Linear Extrapolative Expectations</u>

If (as seems likely) labor income is expected from the viewpoint of a given t to grow at a constant percentage rate, (41) is replaced by

$$y_t(\tau) = \tilde{a}_t e^{\tilde{b}_t(\tau-t)} \qquad (\tau=t+1,\ t+2,\ \ldots)\ , \tag{54.1}$$

in which

$$\tilde{a}_t = \exp[\underline{W}_1' \ln Q_t(\underline{y})]\ , \tag{54.2}$$

and

$$\tilde{b}_t = \underline{W}_2' \ln Q_t(\underline{y})\ . \tag{54.3}$$

The present value $L(\dot{y})$ of expected future changes in labor income as seen from t would be

$$\begin{aligned} L_t &= L\left[\frac{d}{d\tau}\left(\tilde{a}_t\ e^{\tilde{b}_t(\tau-t)}\right)\right] \\ &= L\left[\tilde{a}_t\ \tilde{b}_t\ e^{\tilde{b}_t(\tau-t)}\right] = \tilde{a}_t\tilde{b}_t\int_0^\infty e^{(\tilde{b}_t-\rho)\tau'}d\tau'\ e^{-\tilde{b}_t t} \\ &= \tilde{a}_t\tilde{b}_t\ \frac{1}{\rho-\tilde{b}_t}\ e^{-\tilde{b}_t t} \end{aligned} \tag{55}$$

where the discounting has been carried out on a continuous basis. The present value L_t is finite if (and only if) the rate of discount exceeds the expected rate of growth in labor income.

Since ρ is not known before the estimation of the consumption function, (55) does not provide an operational route for the construction of a unique series on L_t. However, an iterative procedure is available. Given a value of ρ , the series L_t of (55) may be constructed and substituted into (3) _via_ (2). The resulting estimating equation is

$e^{-\tilde{b}_t t}$

$$v_t = (1-\mu) \sum_{j=1}^{n} \gamma_j p_{jt} + \mu[(x_t + \tilde{a}_t \tilde{b}_t) / (\rho - \tilde{b}_t)] \qquad (56)$$

$$+ \underline{i}' \underline{\varepsilon}_t \qquad (t = t^* + 1, \ldots, T) .$$

For any particular setting of ρ , $(T-t^*)$ "observations" on z_t may be generated as

$e^{-\tilde{b}_t t}$

$$z_t(\rho) = [(x_t + \tilde{a}_t \tilde{b}_t) / (\rho - \tilde{b}_t)] . \qquad (57)$$

Equation (56) can then be fitted by OLS. An estimate of δ conditional on a particular setting of ρ (say ρ') is obtained as

$$(\text{estimate of } \delta \mid \rho = \rho') \qquad (58)$$

$$= \hat{\mu} / \rho' \quad ,$$

where $\hat{\mu}$ is the OLSE from (56).

Define the residuals from the fitted version of (56) as

$$\hat{e}_{\rho t} = v_t - \hat{v}_{\rho t} \qquad (59)$$

where $\hat{v}_{\rho t}$ is the fitted value of v at the t^{th} observation, conditional on the particular setting of ρ . Let $\hat{\sigma}^2_\rho$ be the conditional MLE of the variance of e_t ; namely,

$$\hat{\sigma}^2_\rho = (T^*)^{-1} \, \underline{\hat{e}}'_\rho \, \underline{\hat{e}}_\rho \quad , \qquad (60)$$

where $\underline{\hat{e}}_\rho$ is the $(T^* \times 1)$ vector on $\hat{e}_{\rho t}$. $(T^* \equiv T - t^*$.)

The logarithmic likelihood, conditional on ρ , after maximization with respect to δ and the γ_j 's , is

$$\ell_\rho = -\frac{T^*}{2}\ln(2\pi) - \frac{T^*}{2}\ln(\hat{\sigma}^2_\rho) - (2)^{-1}(\hat{\sigma}^2_\rho)^{-1} \; \hat{e}'_\rho \, \hat{e}_\rho \; . \tag{61}$$

/The fitted versions of (3) have $T^* = (T-t^*)$ observations, due to the loss of t^* observations in the generation of the first member of the expectational series $\tilde{b}_t$ ./

The "concentrated" version of the expression for the conditional maximum of the likelihood function is obtained by substituting from (60) into (61):

$$\ell_\rho = -\frac{T^*}{2}\ln(2\pi) - \frac{1}{2}\hat{e}'_\rho \, \hat{e}_\rho - \frac{T^*}{2} \; . \tag{62}$$

It is clear that ℓ_ρ declines monitonically with $\hat{e}'_\rho \, \hat{e}_\rho$ -- that is, with the residual sum of squares (SSE). Since the integral in (55) is defined only for $\rho > \tilde{b}_t$, it is appropriate to restrict our attention to the interval on the line to the right of the maximum value observed in the sample for $\tilde{b}_t$. Let

$$b^* = \underset{t\epsilon[t^*+1,\; T]}{\text{Max}} \left\{\tilde{b}_t\right\} \; . \tag{63}$$

By scanning over values of ρ in the interval (b^*, ∞) and plotting the residual sum of squares (SSE) as a function of ρ , the maximum likelihood estimate of ρ may be read as that value yielding the minimum (SSE).1/ The first and second derivatives of the likelihood function with respect to ρ may be evaluated in the neighbourhood of the MLE of ρ by numerical means.

1/ This is the scanning procedure suggested by Dhrymes [ ] in a somewhat different application.

Numerically obtained results of this type can be combined with analytical results to obtain an estimate of the asymptotic variance-covariance matrix for the estimator of the vector $(\hat{\gamma}_1, \ldots, \hat{\gamma}_n; \hat{\delta}; \hat{\rho}; \hat{\sigma}^2)'$ characterizing the consumption function. (Details are left to Appendix A.)

Determination Length of Effective Memory

The value of t^* remains, at this juncture, unspecified. Since the number of usuable data points diminishes as t^* increases, the maximum likelihood principle cannot be used for its determination as a matter of principle. For clearly, maximization of the likelihood function with respect to t^* will result in t^* being made larger and larger until there are zero degrees of freedom and the likelihood becomes a spike located at the parameter values which fit the reduced data set perfectly. What is needed is a means of trading off degrees of freedom against gains in the maximized value of the likelihood function. As in other approaches (such as [ ]) to fitting lag distributions, there does not seem to be any clear-cut solution for estimating the length of the distribution.[1] As in many previous studies, it seems unlikely that there will be any real possibility of discriminating sharply among different potential lengths of the lag distribution on the basis of short time series (< 30 observations, say). In these circumstances an *ad hoc* experimental approach seems appropriate. One such procedure would be the use of the coefficient of multiple determination corrected for degrees of

1/ As a formal matter, Box-Jenkins methods [ ] provide a method for determinating the length of the lag distribution. The time series data available for this study, however, are almost certainly too short for these methods to be entertained.

freedom, $\bar{R}^2$, as a criterion for comparison of different values of t^* . Given the absence of any formal apparatus, investigation of the joint sampling variability of the resultant estimate of t^* with the other parameter estimates seems intractable at this time.1/

Decay in the Memory Process2/

The extrapolative approach to expectations outlined above does not, as presently formulated, catch the idea of the gradual loss of relevance as historical experience recedes into the past. Another way of looking at the matter is as depicted in Figure 1. It is assumed for both part a and part b

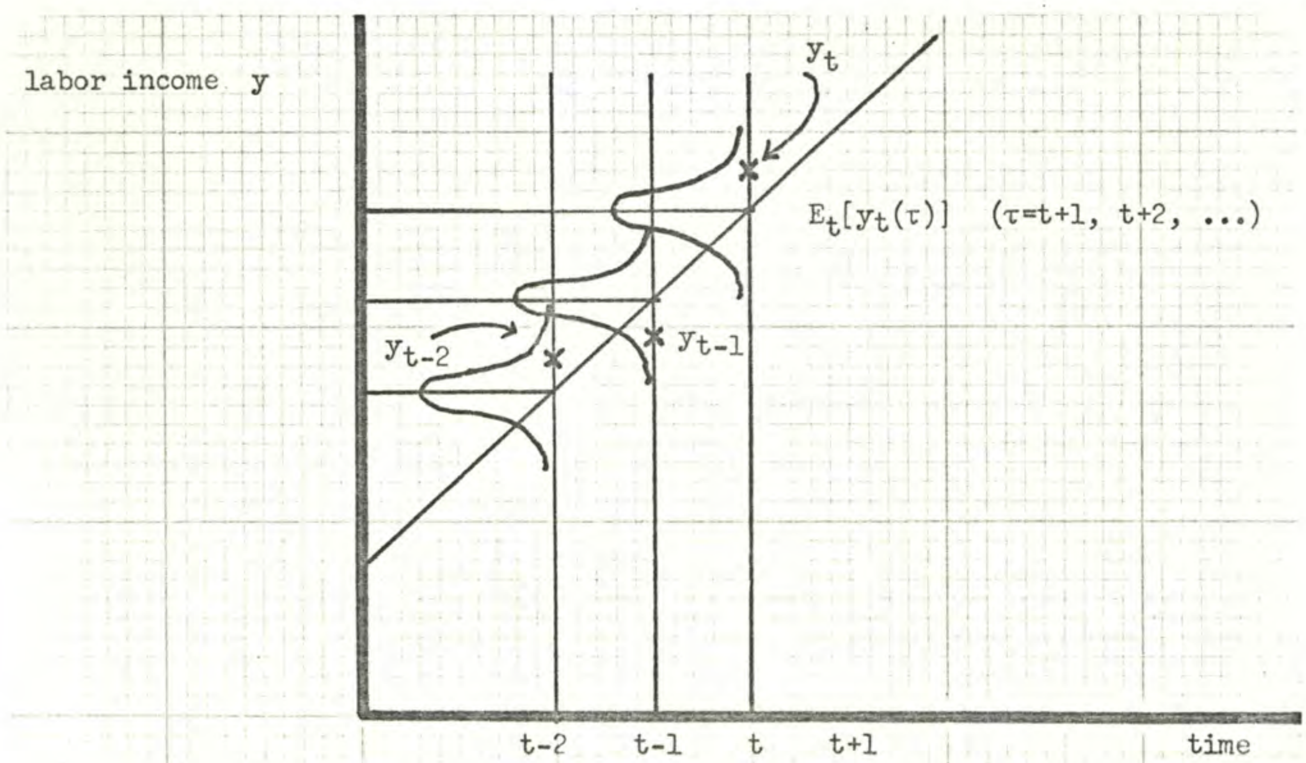


Figure 1a: The line $E_t[y_t(\tau)]$ is the (subjective) conditional expectation of y for future periods as perceived at period t. (In the log-linear extrapolative model, y should be replaced by $\ln y$ throughout.)

1/ Monte-Carlo experiments would be possible, of course, but would be explosive.

2/ This section is written with the linear extrapolative model in kind. The log linear case is covered by writing log y wherever y occurs in this discussion.

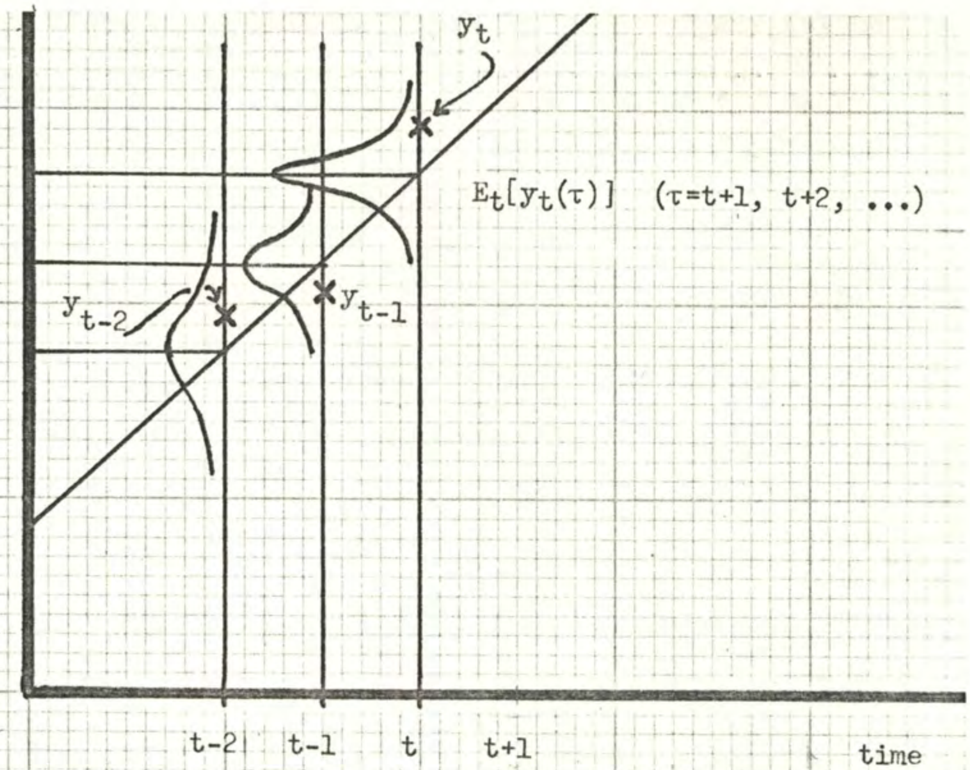


Figure 1b: The line $E_t[y_t(\tau)]$ is the (subjective) conditional expectation of y for future periods as perceived at period t. (In the log-linear extrapolative model, y should be replaced by $\ln y$ throughout.)

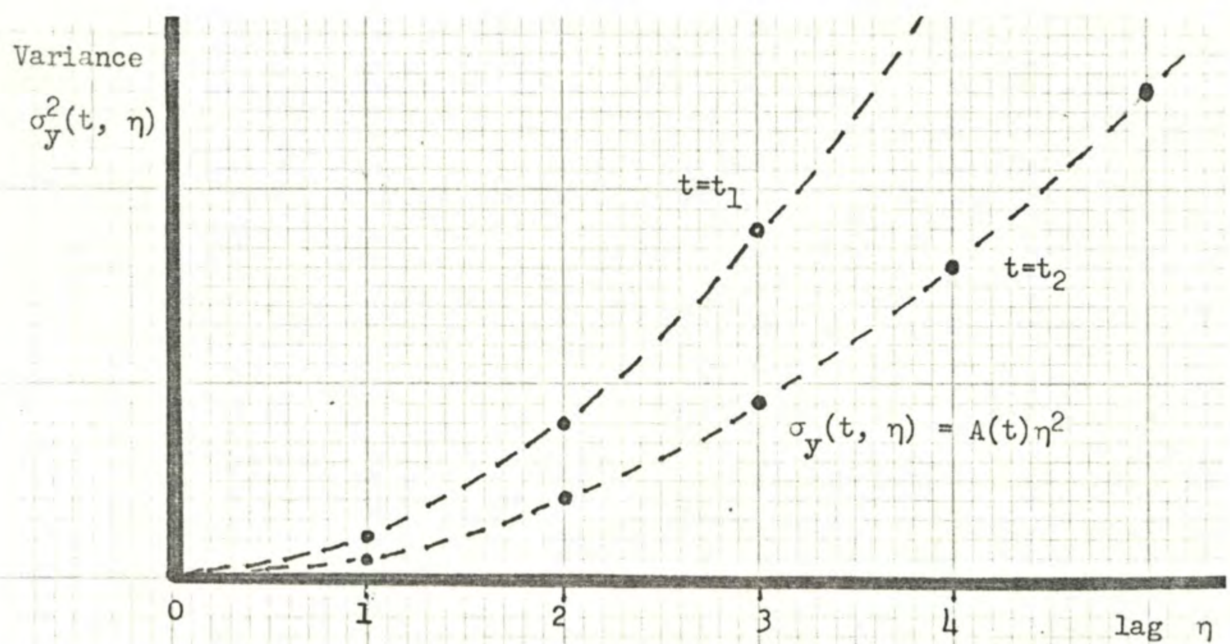


Figure 2: Specification of Functional Dependence of Variance on Lag

of Figure 1 that $t^*=3$. The OLS weight vectors (42.2) and (42.3) may be interpreted as coming from a regime in which the potential information content of each observation y_t, y_{t-1}, y_{t-2} is regarded as the same. Statistically this is captured by the homoscedasticity of the observed y's about their conditional means (as in Figure 1a). These distributions are given a Bayesian interpretation so that they change as the viewpoint t from which expectations are being formulated changes.

From the viewpoint t it is reasonable to regard y_{t-2} as potentially less informative than y_{t-1}. This idea is captured by attaching a larger variance to the distribution of y_{t-2} about its conditional mean than to the distribution of y_{t-1}. [Of course, all of these subjectively viewed variances change when the instant of expectations formation is translated along the time axis through (t+1), (t+2), etc. What is potentially the most informative observation from decision viewpoint t becomes the least potentially informative from decision viewpoint $(t+t^*)$.]

Statistically, this memory decay is modelled by a heteroscedastic error specification (as in Figure 1b). The weight vectors analogous to (42.2) and (42.3) are then determined by generalized least squares [ ]. To make this operationally feasible it is first necessary to specify the functional relationship between the variances of the y values occurring up to t^* periods ago as a function of the lag $(\lambda=1, 2, \ldots, t^*)$. A straight line law of growth in the variances, as in Figure 2, is one simple option. By $\sigma_y^2(t, \lambda)$ denote the variance of $y_{t-\lambda}$ as seen from t for the purpose forming expectations for t, (t+1), etc. The hypothesis of Figure 2 is that

$$\sigma_y^2(t, \lambda) = A(t)\lambda^2 \quad . \tag{64}$$

Assuming freedom from serial correlation in the deviations of the actual y's from their conditional expected means $E_t(y_\tau)$, the generalized least squares estimator of the trend intercept becomes (see [ ], p.)

$$\tilde{b}_t = -\sum_{\lambda=0}^{t^*-1}\left\{y_{t-\lambda}(\lambda-\bar{\lambda})[\sigma_y^2(t, \lambda)]^{-1}\right\}\Big/\left\{\sum_{\lambda=0}^{t^*-1}(\lambda-\bar{\lambda})^2[\sigma_y^2(t, \lambda)]^{-1}\right\} \tag{65.1}$$

and

$$\tilde{a}_t = (t^*)^{-1}\sum_{\lambda=0}^{t^*-1} y_{t-\lambda} - \hat{b}_t\,\bar{\lambda} \tag{65.2}$$

where

$$\bar{\lambda} = (t^* + 1)/2 \quad . \tag{65.3}$$

Weights analogous to (42.2) and (42.3) may be found by substituting from (64) into (65.1) and (65.2). Because generalized least squares depends only on the <u>ratio</u> of variances, the parameters $A(t)$ do not enter into the weight functions. Although (64) is arbitrary, it introduces the notion of memory decay in a plausible form without introducing additional parameters.

January, 1972

Preliminary Draft
Comments Solicited
Not to be Quoted

SAVINGS AND CONSUMER CHOICE

by

Constantino Lluch*

*The author is an economist at the Development Research Center, World Bank. He is very grateful to Alan Powell, who corrected errors in previous work, suggested the approach in section 3 and, by insisting in the importance of "dynamics" in the theory of demand, provided stimulus for section 4. He is also indebted to Louis Phlips, for long discussions on the Houthakker-Taylor model and extensions. The World Bank is not responsible for any views expressed in the paper. All errors are the author's own doing.

CONTENTS

1. Introduction

The aim of this paper is to derive systems of demand equations and their associated consumption functions, from formulations of the consumer problem such that decisions on how much to save and how to allocate expenditure among commodities are treated simultaneously.

The general formulation was presented in [6]. A summary of it is given in section 2.1. In section 2.2, results are given for the case of expected inflation. In section 2.3, the analysis is applied to forms of the utility function commonly encountered in applied demand theory. In section 3, the consequences of exponential trends in the γ-parameters of the Stone-Geary function are analyzed. In section 4 results are given when "consumption capital" is an argument in the utility function. The analysis here is particularly relevant to put in perspective the work by Houthakker and Taylor ([5], Ch. 5).

2. The consumer problem, with $u=u(q)$

2.1 Stationary price expectations

Let us assume that the consumer problem may be formulated as

$$
(A)\left\{\begin{array}{l}
\text{Choose } \; q(t), \;\; 0 \leq t \leq \infty, \\
\text{such that} \\
\qquad \displaystyle\int_0^{\infty} e^{-\delta t}\, u\,[q(t)]\,dt \\
\text{is maximized, subject to} \\
\qquad \dot{w}(t) = \rho w(t) + y(t) - p' q(t), \;\; w(0) = w, \\
\text{and given } \; (\delta, \rho, p, y(t)).
\end{array}\right.
$$

The following notation has been used:

$q(t)$, n-vector of commodity flows;

p , n-vector of prices of commodities in terms of the numeraire stock, $w(t)$;

$w(t)$, wealth at time t;

$y(t)$, exogenous flow of labor income;

ρ , rate of reproduction of wealth;

δ , subjective rate of discount;

$u(\cdot)$, instantaneous utility function.

Notice that when plan (A) is formulated at t=0, the consumer expects (p, ρ) to continue at present levels forever. Expectations about inflation - a uniform rate of increase in all prices - will be introduced later.

The Hamiltonian for problem (A) is

$$H(w(t),q(t),\lambda(t)) = e^{-\delta t} u(q(t)) + \lambda(t)(\rho w(t) + y(t) - p'q(t)),$$

and the set of necessary conditions for an optimal plan $q(t)$ to exist are

(a) $u_q[q(t)] = \lambda(t)\, e^{\delta t}\, p, \qquad 0 \leq t \leq \infty$,

(b) $\dot{w}(t) = \rho w(t) + y(t) - p'q(t), \quad w(0) = w$,

(c) $\dot{\lambda}(t) = -\rho\lambda(t), \quad \lambda(0) = \lambda$,

where u_q is an n-vector of marginal utilities, and $\lambda(t)$ is the implicit utility valuation of savings at time t, discounted to the beginning of the plan.1/

1/ In what follows, it is assumed that an optimum plan satisfying the transversality condition

$$\lim_{t \to \infty} \lambda(t)\, w(t) = 0$$

does exist. Also, inequality constraints are ignored.

Equation (c) can be solved, and the solution put into (a). Also, the flow constraint (b) can be written in integral form. With these changes, the necessary conditions are

$$u_q(q(t)) = \lambda e^{(\delta-\rho)t}\, p, \qquad 0 \leq t \leq \infty \tag{1}$$

$$w + L(y) = p'L(q) \tag{2}$$

where

$$L(y) = \int_0^\infty e^{-\rho t}\, y(t)dt, \qquad L(q) = \int_0^\infty e^{-\rho t}\, q(t)dt.$$

The relationship between (1), (2) and the set of necessary conditions for the problem "max $u(q)$ subject to $p'q = v$" comes sharply to focus when system (1) is written for $t=0$ and the present value constraint (2) is written in terms of $q(0)$, $y(0)$.

Let

$$q(0) = q, \quad y(0) = y.$$

Then, using the fact that

$$L(\dot{y}) = \rho L(y) - y, \quad L(\dot{q}) = \rho L(q) - q,$$

the set of necessary conditions at the beginning of the plan is

$$u_q(q) = \lambda p, \tag{3}$$

$$p'q = z - p'L(\dot{q}) \;, \tag{4}$$

where

$$z = \rho w + y + L(\dot{y}) \;.$$

System (3) is identical with the set of marginal conditions in static demand theory. The "budget constraint" (4) contains the difference between

(A) and the usual formulation. In fact, (A) reduces to the usual problem if $\dot{q} = \dot{y} = 0$, - stationarity in the optimal consumption plan and expected labor income flow.

Let us assume that the consumer replans continuously. Then, his behavior over time is given by successive solutions of (3), (4): real time is seen as a continuum of (t=0)'s, beginnings of plans. At each instant, the solutions

$$q(p,\ z;\ p'L(\dot{q})), \qquad v = v(p;\ z;\ p'L(\dot{q})),$$

where $v \equiv p'q$, represent the system of commodity demand equations and the consumption function.

Notice that the consumption function is given by (4) once (i) expectations on labor income flows are made explicit, i.e., the $L(\dot{y})$ component of z is specified; (ii) the impact of the plan on present behavior through $p'L(\dot{q})$ is taken into account. To evaluate $p'L(\dot{q})$, the system $\dot{q}(t)$ is crucial. This "basic system of differential equations in the theory of demand" is obtained from (1),

$$\dot{q}(t) = \lambda(\delta-\rho)e^{(\delta-\rho)t}\, U^{-1}(t)p, \quad 0 \leq t \leq \infty, \qquad (5)$$

where $U(t)$ is the Hessian of the utility function at time t, as seen at t=0.

2.2 Expected inflation

If prices are expected to go up over time at the rate π, the expected current account of the consumer (the differential equation $\dot{w}(t)$), has to be modified. Let us assume that inflation yields capital gains, so that income from non-human wealth is measured as $(\rho + \pi)w(t)$. Then, the basic flow constraint under expected inflation is

$$\dot{w}(t) = (\rho+\pi)w(t) + y(t) - e^{\pi t}p'q(t), \qquad w(0) = w.$$

Using this constraint in problem (A), it can be seen that system (1) remains unchanged, and the present value constraint (2) is rewritten as

$$w + M(y) = p'L(q),$$

where

$$M(y) = \int_0^{\infty} e^{-(\rho+\pi)} y(t)dt.$$

Thus, the nominal expected income flow $y(t)$ is discounted at the nominal rate of interest, $(\rho+\pi)$, while consumption expenditures flows are discounted at the real rate, ρ . Equations (3) and (4) are written as

$$u_q(q) = \lambda p \ , \tag{6}$$

$$p'q = \hat{z} - p'L(\dot{q}), \tag{7}$$

where

$$\hat{z} = \frac{\rho}{\rho+\pi}\left\{y + (\rho+\pi)w + M(\dot{y})\right\} .$$

The relevance of expected inflation is thus contained in the "income" variable, $\hat{z}$. Notice that $\hat{z} \to z$ as $\pi \to 0$.

2.3 Applications

A. The Extended Linear Expenditure System, ELES.

Let

$$u(q(t)) = \beta' \log\,(q(t)-\gamma), \qquad i'\beta = 1,$$

where $(\beta,\ \gamma)$ are n-vectors of parameters and i is a vector of unit elements. Then we can derive[2]

$$u_q(q(t)) = (\hat{q}(t)-\hat{\gamma})^{-1}\ \beta,$$

$$\dot{q}(t) = (\rho-\delta)(q(t)-\gamma),$$

$$p'L(\dot{q}) = \left(\frac{\rho}{\delta} - 1\right) p'(q-\gamma).$$

Therefore, equations (3) and (4) can be written as

$$(\hat{q} - \hat{\gamma})^{-1}\ \beta = \lambda p, \tag{8}$$

$$p'q = (1-\mu)p'\gamma + \mu z, \qquad \mu = \delta/\rho\ , \tag{9}$$

a system of $(n+1)$ equations in $(q,\ \lambda)$. The solution for q yields

$$\hat{p}q = \hat{p}\gamma + \mu\beta(z - p'\gamma)\ . \tag{10}$$

Adding up the equations in (10), and given $i'\beta=1$, the consumption function (9) is obtained.

Define $p'q=v$. Substituting z from (9), into (10), we get

$$\hat{p}q = \hat{p}\gamma + \beta(v - p'\gamma)$$

which is the Linear Expenditure System (LES), widely used in applied demand

---

2/ The symbol (^) over a vector denotes the diagonal matrix formed with the elements of that vector.

theory.[3] Therefore, it is natural to name ELES the system of (n+1) equations

$$\hat{p}q = \hat{p}\gamma + \beta(v - p'\gamma), \tag{11}$$

$$v = (1-\mu)p'\gamma + \mu z , \tag{12}$$

relation (12) being the consumption function associated with Stone's LES, (11).

The only change needed to incorporate expectations of a constant inflation rate into ELES is to write $\hat{z}$ instead of z in (12).

B. The Extended Generalized Linear Expenditure System, EGLES.

Let

$$u(q(t)) = \alpha^{-1} \sum \beta_i (q_i - \gamma_i)^{\alpha} ,$$

which tends to the log linear function in the previous section as $\alpha \to 0$. The scalar α increases substitution possibilities in the solution to the consumer problem.[4] Then we can derive

$$u_i(q(t)) = (q_i(t) - \gamma_i)^{\alpha-1} \beta_i , \quad (i=1,\ldots, n)$$

$$\dot{q}(t) = (\delta-\rho)(\alpha-1)^{-1} (q(t)-\gamma),$$

$$p'L(\dot{q}) = (\delta-\rho)(\rho\alpha-\delta) p'(q-\gamma).$$

The notation $u_i \equiv \partial u/\partial q_i$ has been used. Therefore, equations (3) and (4)

3/ See, for example, Stone [12], Parks [7], Goldberger and Gamaletsos [4], and Solari [10].

4/ EGLES has been recently proposed and estimated by Solari [11]. Notice that the resulting expenditure equations are not linear in prices.

can be written as

$$(q_i - \gamma_i)^{\alpha-1} \beta_i = \lambda p_i, \qquad (i=1,\ldots, n) \tag{13}$$

$$(\alpha - 1)p'q = (\mu-1)p'\gamma + (\alpha-\mu)z , \tag{14}$$

a system of (n+1) equations in (q, λ). The solution for q yields

$$\hat{p}q = \hat{p}\gamma + \mu^*\beta^*(z - p'\gamma) \tag{15}$$

where the scalar μ^* and the ith element of the vector β^* are defined as

$$\mu^* = (\alpha - \mu)(\alpha - 1)^{-1}$$

$$\beta_i^* = \frac{p_i^{2-\alpha} \beta_i^{\alpha-1}}{\sum_i p_i^{2-\alpha} \beta_i^{\alpha-1}} , \qquad i=1,\ldots, n.$$

Adding up the equations in (15), and given $i'\beta^*=1$, we obtain the consumption function (14).

Again, let $p'q \equiv v$. Substituting z from (14) into (15) we get

$$\hat{p}q = \hat{p}\gamma + \beta^*(v - p'\gamma) ,$$

the system of expenditure equations derived by Solari [11]. Therefore, it is natural to name EGLES the system of (n+1) equations

$$\hat{p}q = \hat{p}\gamma + \beta^*(v - p'\gamma) , \tag{16}$$

$$v = (1-\mu^*)p'\gamma + \mu^*z . \tag{17}$$

The notational similarity between (16) and (11) hides a basic difference: β^* is a function of prices; β is a vector of parameters.

To incorporate into EGLES expectations of a constant inflation rate it suffices to write $\hat{z}$ instead of z in (17).

C. The Extended Direct Addilog System, EDAS.

Let

$$u(q(t)) = \sum_i \beta_i q_i(t)^{\alpha_i} \quad .$$

From this specification of the utility function we can derive

$$u_i(q_i(t)) = \alpha_i \beta_i q_i(t)^{\beta_i - 1}, \qquad (i=1,\ldots, n),$$

$$\dot{q}_i(t) = (\delta - \rho)(\beta_i - 1)^{-1} q_i(t) \ ,$$

$$p' L(\dot{q}) = (\mu - 1) \sum (\beta_i - \mu)^{-1} p_i q_i \quad .$$

Therefore, equations (3) and (4) can be written as

$$\alpha_i \beta_i q_i^{\beta_i - 1} = \lambda p_i \ , \qquad (i=1,\ldots, n) \ , \tag{18}$$

$$\sum (\beta_i - 1)(\beta_i - \mu)^{-1} p_i q_i = z \quad , \tag{19}$$

a system of (n+1) equations in (q, λ). The multiplier λ cannot be eliminated in a simple manner, as in previous sections, due to nonlinearity in (18). Let us introduce some additional notation:

$$p_i^* = (p_i / \alpha_i \beta_i)^{\frac{1}{\beta_i - 1}} ; \ \Lambda(\lambda) = \mathrm{diag}(\lambda^{\frac{1}{\beta_i - 1}}) \ ; \ \ \Omega = \mathrm{diag}(\beta_i - 1)(\beta_i - \mu)^{-1} \ ,$$

so that $\Lambda(\lambda)$, Ω are diagonal matrices with typical elements $\lambda^{\frac{1}{\beta_i - 1}}$, $(\beta_i - 1)(\beta_i - \mu)^{-1}$, respectively, and p^* is an n-vector with typical element p_i^*, $(i=1,\ldots, n)$.

Then, systems (18), (19) can be written in matrix notation

$$\Lambda(\lambda) q = p^* \ , \ \ p' \Omega q = z \ ,$$

and the solution for λ is given implicitly by

$$p'\Omega\Lambda(\lambda)^{-1}p^* = z$$

Denoting by λ_* the solution $\lambda(z, p)$ in this expression, and letting $\Lambda_* =$ diag$(\lambda_*^{\beta i-1})$ the commodity demand equations and the aggregate consumption function that constitute the extended direct addilog system, (EDAS), are

$$q = \Lambda_*^{-1}\, p^* \;, \tag{20}$$

$$p'q = p'\Lambda_*^{-1}\, p^* \;. \tag{21}$$

Notice that it is not possible to write q in (20) as a function of $v \equiv p'q$ only (by substituting z from (21) into (20), as it was done in previous sections). The implication is that, in the absence of savings, the systematic part of the direct addilog system as formulated in the in the literature[5] is misspecified.

To incorporate inflationary expectations into EDAS, it suffices to write $\hat{z}$ instead of z in (20),(21).

D. The Extended Quadratic Utility System, (EQUS).

Let

$$u(q(t)) = aq(t) - \frac{1}{2}\, q(t)'Aq(t) \;,$$

where (a, A) are an n-vector and a (nxn) matrix of parameters. Then we can derive

$$u_q(q(t)) = a - Aq(t) \;,$$

$$\dot{q}(t) = (\delta-\rho)(q(t) - A^{-1}a) \;,$$

5/ See Goldberger [3], pp. 81-84, and references cited there.

$$p'L(\dot{q}) - (\mu-1)(2-\mu)^{-1}\, p'(q - A^{-1}a)\ .$$

Therefore, equations (3) and (4) can be written as

$$a - Aq = \lambda p \tag{22}$$

$$p'q = (\mu-1)p'A^{-1}\, a + (2-\mu)z\ , \tag{23}$$

a system of (n+1) equations in (q, λ). The solution for q yields

$$q = A^{-1}a + \frac{2-\mu}{\mu-1}\,\frac{z-p'q}{p'A^{-1}p}\,A^{-1}p\ , \tag{24}$$

the system of commodity demand equations in EQUS. Adding up expenditures in commodities (i.e., premultiplying (24) by p'), we obtain the aggregate consumption function (23).

Substituting z from (23) into (24) we obtain

$$Aq = a - (p'A^{-1}p)^{-1}(v - p'A^{-1}a)p$$

which is the demand system that would result from the set of necessary conditions of static demand theory,

$$a - Aq = \lambda p, \quad p'q = v\ .$$

Therefore, it is natural to name EQUS the system of (n+1) equations

$$q = A^{-1}a - (p'A^{-1}p)^{-1}(v - p'A^{-1}a)A^{-1}p\ , \tag{25}$$

$$v = (\mu-1)p'A^{-1}a + (2-\mu)z\ , \tag{26}$$

relation (26) being the consumption function when the utility function is specified as a quadratic.6/

6/ For the formulation of the system in classical demand theory, and some references to its use, see Goldberger [3], pp. 73-80. The extension by Houthakker and Taylor [5] will be taken up later.

To incorporate expectations of a constant inflation rate into EQUS, it suffices to write $\hat{z}$ instead of z in (26).

3. The consumer problem with $u=u(q, t)$: application to ELES

It is reasonable to assume that the utility function, as seen by the consumer, is not time invariant. The simplest way of taking into account trends in preferences, in the context of a specific utility function (the log linear case) is as follows.

Let

$$u(q(t)) = \beta' \log(q(t) - \gamma(t)) ,$$

such that all the elements in the vector $\gamma(t)$ are shifting at a constant rate, σ ,

$$\dot{\gamma}(t) = \sigma\gamma(t) , \quad \gamma(0) = \gamma .$$

In this case, the Hamiltonian for (A) with this utility function is

$$\begin{aligned} H(w(t), \gamma(t), q(t), \lambda(t), \psi(t)) = \\ = e^{-\delta t} \beta' \log(q(t)-\gamma(t)) + \gamma(t)(\rho w(t)+y(t)-p' q(t)) + \\ + \psi(t)'(\sigma\gamma(t)) , \end{aligned}$$

and the corresponding set of necessary conditions[7/] is

(a) $$(\hat{q}(t) - e^{\sigma t}\hat{\gamma})^{-1}\beta = \gamma e^{(\delta-\rho)t}\, p ,$$

7/ It is assumed that there exist optimal paths such that the transversality conditions

$$\lim_{t \to \infty} \lambda(t)w(t) = \lim_{t \to \infty} \psi_i(t)\gamma_i(t) = 0, \quad (i=1,\ldots, n)$$

are fulfilled.

(b) $\quad w + L(y) = p'L(q)$,

(c) $\quad \dot{\psi}(t) = -\sigma\psi(t) + \lambda e^{-\rho t} p.$

The system (c) can be ignored for present purposes. It establishes the rate of change in the utility valuation of changes in $\gamma(t)$:

$$\frac{\dot{\psi}_i(t)}{\psi_i(t)} = -\sigma + \frac{\lambda(t)}{\psi_i(t)} p_i \ , \quad (i=1,\ldots, n).$$

Let us focus attention on (a) and (b). From (a) we can derive

$$\dot{q}(t) = (\rho-\delta)(q(t) - e^{\sigma t}\gamma) + \sigma e^{\sigma t}\gamma \ ,$$

$$p'L(\dot{q}) = (\frac{\rho}{\delta} - 1)p'q - \frac{\rho-\delta+\sigma}{\rho-\sigma} \ \frac{\rho}{\delta} p'\gamma \ .$$

Therefore, equations (3) and (4) can be written now as

$$(\hat{q} - \hat{\gamma})^{-1} \beta = \lambda p \ , \tag{27}$$

$$p'q = \mu z + \nu p'\gamma \ , \qquad \nu = (\rho-\delta+\sigma)(\rho-\sigma)^{-1} \tag{28}$$

a system of (n+1) equations in (q, λ). The solution for q yields

$$\hat{p}q = \hat{p}\gamma + \beta(\mu z + (\nu-1)p'\gamma) \ , \tag{29}$$

the expenditure system associated with problem (A) when the utility function is log linear and the $\gamma_i(t)$'s $(i=1,\ldots,n)$ are expected to grow exponentially at the rate σ . Adding up the expenditure equations (29), the consumption function (28) is obtained. Let $p'q \equiv v$ in (28). Substitute z from (28) into (29), to get

$$\hat{p}q = \hat{p}\gamma + \beta(v - p'\gamma) \ ,$$

the linear expenditure system as usually estimated. Therefore the system

$$\hat{p}q = \hat{p}\gamma + \beta(v - p'\gamma) , \tag{30}$$

$$v = \nu p'\gamma + \mu z , \tag{31}$$

is the ELES with exponential trend in γ .8/

As before, substitution of $\hat{z}$ for z introduces a constant expected inflation rate into the analysis.

4. The consumer problem, with $u=u(q, s)$

4.1 The role of consumption capital

Attempts have been made to formalize the notion that utility today is not independent of past consumption experience.9/ These attempts are of particular importance to remove in part the consequences of intertemporal additivity in problem (A).

Let us postulate a utility functional of the form

$$U(q, s) = \int_0^{\infty} e^{-\delta t}\, u(q(t), s(t))dt$$

8/ A warning is in order: given that $\gamma=\gamma(0)$, the vector γ should in principle be dated, in empirical applications. To keep it as a vector of estimated parameters, additional assumptions are needed (for example perfect foresight of consumers with respect to $\gamma(t)$.

9/ The variable s in Houthakker and Taylor [5], p. 10, stands for habit formation and/or durable goods stocks. The second interpretation has to dropped (Pollak [9], p. 76), as wealth portfolio problems cannot be accommodated into their formulation. The first interpretation can be retained, but their analysis is not satisfactory, as they treat what is basically a problem in calculus of variations with techniques of differential calculus. Phlips [8] has considered a model basically identical to the one analyzed here. The differences between both treatments are: the insistence here in the distinction between subjective, planning time and real time, which implies that behavior at $t=0$ is the main concern (assuming that an optimal consumption path does exist); and the belief here that this formulation cannot yet be applied to durable goods, (for the same reason that the Houthakker and Taylor formulation is not applicable either).
Von Weizsacker analyzed in [13] the impact of past consumption on today's preferences.

where the n-vector $s(t)$ is defined by

$$\dot{s}(t) = q(t), \quad s(0) = s ,$$

i.e., $s(t)$ is the vector of accumulated consumption of commodities, (vector of consumption capital), at time t.[10/]

The consumer problem might be formulated as

$$(B)\quad \begin{cases} \text{Choose} \quad q(t), \quad 0 \leq t \leq \infty , \\ \text{such that} \quad U(q, s) \text{ is maximized, subject to} \\ \qquad \dot{w}(t) = \rho w(t) + y(t) - p'q(t), \quad w(0)=w \\ \qquad \dot{s}(t) = q(t), \quad s(0)=s , \\ \text{and given} \quad (\rho, \delta, y(t), p) . \end{cases}$$

The Hamiltonian for this problem is

$$H(w(t), s(t), q(t), \lambda(t), \psi(t) =$$
$$e^{-\sigma t}\, u(q(t),s(t)) + \lambda(t)(\rho w(t) + y(t) - p'q(t)) + \psi(t)'q(t) ,$$

For the path $q(t)$, $0 \leq t \leq \infty$, to be optimum in problem (B), a scalar multiplier $\lambda(t)$ and a vector multiplier $\psi(t)$ must exist such that

---

10/ It is assumed that consumption capital does not depreciate, i.e., there is no "memory loss". To accommodate a constant "memory loss" rate is very simple. The time path of consumption capital would be given by

$$\dot{s}(t) = q(t) - \alpha s(t), \; s(0)=s ,$$

where α is the rate of "memory loss". (Unfortunately, the symbol α has been used before, section 2.3.B, with entirely different meaning.)

$$\text{(a)} \qquad e^{-\delta t}\, u_q(t) + \psi(t) = \lambda(t)p, \quad 0 \leq t \leq \infty ,$$

$$\text{(b)} \qquad w + L(y) = p'L(q) ,$$

$$\text{(c)} \qquad s = \rho L(s) - L(q) , \tag{32}$$

$$\text{(d)} \qquad \dot{\lambda}(t) = -\rho\lambda(t) , \quad \lambda(0)=\lambda ,$$

$$\text{(e)} \qquad \dot{\psi}(t) = -e^{-\delta t} u_s(t) , \quad \psi(0)=\psi ,$$

where $u_q(t) \equiv u_q(q(t),s(t)), u_s(t) = u_s(q(t),s(t))$, the vectors of marginal utility.11/

It is important to give an economic, intuitively appealing interpretation to (a)-(e), given that the presence of consumption capital in the utility function changes the consumer problem substantially.

System (32a) expresses the fact that any consumption plan is optimal if and only if marginal utility (MU) per unit of expenditure at time t, discounted to $t=0$, is the same for all commodities. This MU has two components: the discounted direct MU from consumption, $e^{-\delta t}\, u_q$, and the discounted implicit utility valuation of additions to consumption capital, $\psi(t)$. In an

---

11/Again, it is assumed that there exist optimal paths such that the transversality conditions

$$\lim_{t \to \infty} \lambda(t)w(t) = \lim_{t \to \infty} \psi_i(t)s_i(t)=0, \quad (i=1,\ldots,n)$$

are fulfilled. If there is "memory loss", (32c) must be rewritten as

$$s = (\rho+\alpha)L(s) - L(q) , \tag{32c'}$$

and (32e) as

$$\dot{\psi}(t) = \alpha\psi(t) - e^{-\delta t}u_s(t), \quad \psi(0)=\psi . \tag{32e'}$$

optimal plan the total MU is equal to the discounted implicit valuation of savings at time t, $\lambda(t)$.

Equation (32b) is the familiar wealth constraint.

Equation (32c) is a reformulation of the definition of consumption capital as the integral of consumption flows. The present value of consumption flows associated with the plan $L(q)$, equals the present value of changes in consumption capital, $L(\dot{s})$. By definition, $L(\dot{s}) = \rho L(s) - s$.

Equation (32d) gives the rule governing the change over time in the implicit discounted utility valuation of savings at t, $\lambda(t)$. It declines exponentially at the rate of reproduction of wealth, ρ .

System (32e) gives the rule governing the change over time in the implicit discounted utility valuation of increments to consumption capital, $\psi(t)$. Its time derivative is negative, with absolute value equal to the discounted direct marginal utility of consumption capital. System (e) has a clear interpretation when written in integral form.[12]

$$\psi(t) = \int_t^{\infty} e^{-\delta t}\, u_s(\tau)d\tau \qquad (32e)$$

$$\equiv \bar{N}_t(u_s)$$

i.e., along an optimal plan, $\psi(t)$ must be equal to the direct utility brought

---

12/ If there is "memory loss", the integral form of (32e) is

$$\psi(t) = e^{\alpha t}\int_t^{\infty} e^{-(\delta+\alpha)\tau}\, u_s(\tau)d\tau$$

$$\equiv e^{\alpha t}\, N_t(u_s) \ .$$

about by the increment in consumption capital for all future times. This direct utility is the integral in (32e), defined as $\bar{N}_t(u_s)$ for notational purposes.

Substituting the solutions to (32d) and (32e) into (32a) we can write (32) as follows:

$$\begin{aligned}
&(a) \qquad u_q(t) + e^{\delta t}\bar{N}_t(u_s) = \lambda e^{(\delta-\rho)t}p, \qquad 0 \leq t \leq \infty ,\\
&(b) \qquad w + L(y) = p'L(q) , \qquad\qquad (33)\\
&(c) \qquad s = \rho L(s) - L(q) .
\end{aligned}$$

System (33) has (2n+1) equations to be solved for $(q(t), s(t), \lambda)$, given w, s - initial non-human wealth and consumption capital. This system is the extension of (1), (2) due to the presence of consumption capital in the utility function. The relationship between (33) and the set of necessary conditions for the problem

$$\max u(q) \quad \text{subject to} \quad p'q = v$$

and for problem (A) comes sharply to focus by writing the marginal conditions (33a) at t=0, and by making initial consumption, consumption capital and income, (q, s, y), appear explicitly. Then, the necessary conditions for problem (B) at t=0 are

$$\begin{aligned}
&(a) \qquad u_q(q, s) + \psi = \lambda p ,\\
&(b) \qquad p'q = z - p'L(\dot{q}) , \qquad\qquad (34)\\
&(c) \qquad q + \rho s = \rho^2 L(s) - L(\dot{q}) ,
\end{aligned}$$

where[13/]

$$\psi = \bar{N}(u_s) = \int_0^\infty e^{-\delta t} u_s(t)dt \ .$$

Given $(L(s), L(\dot{q}), z, p, s)$, system (34) has $(2n+1)$ equations in $(2n+1)$ variables (q, ψ, λ). Under the assumption of continuous replanning by the consumer, his behavior over time can be represented by successive solutions of (34).

Thus, there are two main differences between (34) and the necessary conditions (3), (4) for problem (A): (i) the presence of ψ, the implicit discounted utility valuation of additions to consumption capital at $t=0$, in the marginal condition (34a); (ii) the definition (34c), relating initial consumption and consumption capital, (q, s), given an optimal plan $q(t)$.

An explicit solution to the problem cannot be found unless $L(\dot{q})$, $L(s)$ are specified. The "basic system of differential equations in the theory of demand with consumption capital in the utility function" is obtained by differentiating equation (33a):

$$\dot{q}(t) = u_{qq}^{-1}(t)\left\{u_s(t) + \delta u_q(t) - u_{qs}(t)q(t) - \lambda \rho e^{(\delta-\rho)t} p\right\} \qquad (35)$$

$$0 \leq t \leq \infty$$

where (u_{qq}, u_{qs}) are square matrices with typical elements

---

13/ If there is "memory loss",

$$\psi = N(u_s) = \int_0^\infty e^{-(\delta+\alpha)t} u_s(t)dt \ ,$$

and (34c) has to be rewritten as

$$q + \rho s = \rho(\rho+\alpha)L(s) - L(\dot{q}) \ .$$

$$(\partial^2 u/\partial q_i \; \partial q_j) \; , \qquad (\partial^2 u/\partial q_i \; \partial s_j) \; ,$$

$(i, j=1, \dots, n)$, respectively. It can be seen that (35) reduces to (5) if $s(t)$ is not an argument in the utility function.[14]

To proceed, it is convenient to specify the utility function, so that $L(s)$, $L(\dot{q})$ can be written in terms of (q, s, λ).

Expected inflation is again accounted for by writing $\hat{z}$ instead of z.

4.2 Applications

A. EQUS with consumption capital

Let

$$u(t) = a'q + b's + \frac{1}{2} q'Aq + q'Bs + \frac{1}{2} s'Cs$$

where $q(t)$, $s(t)$ are functions of time, (a, b) are n-vectors, (A, B, C) are diagonal matrices of constant parameters. Then we can derive, (assuming that consumption capital vanishes at the rate α),

$$u_q(t) = a + Aq(t) + Bs(t) \; ,$$

$$u_s(t) = b + Cs(t) + Bq(t) \; ,$$

$$u_{qq} = A \; , \qquad u_{qs} = B \; ,$$

$$\dot{q}(t) = (\delta+\alpha)q(t) + Ds(t) - \lambda(\alpha+\rho)e^{(\delta-\rho)t} A^{-1} p + c$$

14/ If there is "memory loss", the basic differential equation is

$$\dot{q}(t) = u_{qq}^{-1} \Big\{ u_s(t) + (\delta+\alpha)u_q(t) - u_{qs}(t)(q(t) - \alpha s(t)) - \lambda(\alpha+\rho)e^{(\delta-\rho)t} p \Big\} \; , \quad 0 \leq t \leq \infty \tag{35'}$$

where $c = A^{-1}[(\delta+\alpha)a+b]$, $D = A^{-1}[(\delta+2\alpha)B+C]$. This differential equation is (35′), given the quadratic utility function specified in this section.

a. Derivation of $L(\dot{q})$, $L(s)$.

If $\delta < 2\rho$, and given

$$L(\dot{s}) = L(q) - \alpha L(s) = \rho L(s) - s$$

we obtain

$$L(\dot{q}) = Eq + Fs + \lambda Gp + d,$$

a linear form in (q, s, λ), i.e., in consumption, consumption capital and implicit utility valuation of savings at the beginning of the plan. The following definitions have been used:

$$E = (I-\mu H)^{-1}\mu H\ , \qquad H = I + [(\delta+\alpha)\rho]^{-1}D, \qquad \mu = (\delta+\alpha)/\rho$$

$$F = \rho^{-1}(I-\mu H)^{-1}D\ ,$$

$$G = (\alpha+\rho)(2\rho-\delta)^{-1}(I-\mu H)^{-1}A^{-1}\ ,$$

$$d = \rho^{-1}(I-\mu H)^{-1}c\ .$$

Next, let us consider the system of 2n differential equations in matrix form

$$\begin{bmatrix}\dot{q}(t)\\ \dot{s}(t)\end{bmatrix} = \begin{bmatrix}(\delta+\alpha)I & D\\ I & -\alpha I\end{bmatrix}\begin{bmatrix}q(t)\\ s(t)\end{bmatrix} + \begin{bmatrix}\phi(t)\\ 0\end{bmatrix}$$

where

$$\phi(t) = c - \lambda(\alpha+\rho)e^{(\delta-\rho)t}A^{-1}p\ .$$

Its solution is [15/]

$$\begin{bmatrix} q(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} D_1(t) & DD_2(t) \\ D_2(t) & D_1(t)-(\delta+2\alpha)D_2(t) \end{bmatrix} \begin{bmatrix} q \\ s \end{bmatrix} + \begin{bmatrix} \int_0^t D_1(t-\tau)\phi(\tau)d\tau \\ \int_0^t D_2(t-\tau)\phi(\tau)d\tau \end{bmatrix}$$

The n-matrix D_1 is diagonal, with typical elements D_1^i , $(i=1,\ldots, n)$, defined by

$$D_1^i = \frac{1}{\Theta_{ii}} \left\{(s_1^i + \alpha)e^{s_1^i t} - (s_2^i + \alpha)e^{s_2^i t}\right\}$$

with
$$\Theta_{ii} = (\delta^2 + 4c_{ii})^{1/2},$$

$$c_{ii} = d_{ii} + \alpha(\delta + \alpha) ,$$

$$s_1^i = \frac{\delta + \Theta_{ii}}{2} ,$$

$$s_2^i = \frac{\delta - \Theta_{ii}}{2} ,$$

and d_{ii} being the ith diagonal element of D. The matrix D_2 is also diagonal, with typical elements D_2^i , $(i=1,\ldots, n)$, defined by

$$D_2^i = \frac{1}{\Theta_{ii}} \left\{ e^{s_1^i t} - e^{s_2^i t} \right\} .$$

15/ Let

$$x(t) = \begin{bmatrix} q(t) \\ s(t) \end{bmatrix} , \quad Q = \begin{bmatrix} (\delta+\alpha)I & D \\ I & -\alpha I \end{bmatrix} , \quad n(t) = \begin{bmatrix} \phi(t) \\ o \end{bmatrix} .$$

15/ The solution of

$$\dot{x}(t) = Qx(t) + \eta(t), \; x(o) = x,$$

is

$$x(t) = e^{Qt} x + \int_o^t e^{Q(t-\tau)} \eta(\tau)\, d\tau \; .$$

The matrix e^{Qt} is found as the inverse Laplace transform of $(sI-Q)^{-1}$, (see Athans and Falb [1], p. 141). Note that

$$(sI-Q) = \begin{bmatrix} (s-\delta')I & -D \\ -I & (s+\alpha)I \end{bmatrix}, \qquad \delta' = \delta+\alpha \; ,$$

$$(sI-Q)^{-1} = \begin{bmatrix} (s-\delta')^{-1}[I + D(\epsilon I-D)^{-1}] & D(\epsilon I-D)^{-1} \\ (\epsilon I - D)^{-1} & (s-\delta')(\epsilon I-D)^{-1} \end{bmatrix},$$

where $\epsilon = (s+\alpha)(s-\delta')$. The assumption that D is diagonal is essential to obtain simple expressions for the elements of this inverse. The typical elements for each of the diagonal submatrices are, (clockwise),

$$\frac{s+\alpha}{\Delta_{ii}}, \; \frac{d_{ii}}{\Delta_{ii}}, \; \frac{s-\delta'}{\Delta_{ii}}, \; \frac{1}{\Delta_{ii}},$$

with $\Delta_{ii} = (s-\delta')(s+\alpha) - d_{ii}$. The inverse Laplace transform is immediate (see Churchill [2], p. 324), writing Δ_{ii} as $(s-s_1^i)(s-s_2^i)$.

In particular, the solution for the vector of consumption capital along an optimal plan is

$$s(t) = D_2(t)[q+(\delta+2\alpha)S + D_1(t)s + \int_0^t D_2(t-\tau)\phi(\tau)d\tau ,$$

an expression that involves only initial values, (q, s, λ). We are interested in obtaining

$$L(s) = \int_0^\infty e^{-\rho t} s(t)dt ,$$

the present value of the elements in the vector of consumption capital along an optimal plan. It is

$$\begin{aligned} L(s) &= L(D_2)\Big[q -(\delta+2\alpha)s\Big] + L(D_1)s + L\Big[\int_0^t D_2(t-\tau)\phi(\tau)d\tau\Big] \\ &= L(D_2)\Big[L(\phi) + q - (\delta+2\alpha)s\Big] + L(D_1)s , \end{aligned}$$

given that[16/]

$$L\Big[\int_0^t D_2(t-\tau)\phi(\tau)\Big] = L(D_2)\ L(\phi) .$$

But, if $\rho > s_j^i$, $(i = 1, \ldots, n;\ j = 1, 2)$, we have

$$L(D_2) = (\rho_1 I-D)^{-1} , \qquad \rho_1 = (\alpha+\rho)(\alpha-\rho-\delta)$$

$$L(D_1) = (\alpha+\rho)(\rho_1 I-D)^{-1} ,$$

$$L(\phi) = \rho^{-1}c - \lambda\rho_2 A^{-1}p , \quad \rho_2 = (\alpha+\rho)(\delta-2\rho)^{-1} .$$

16/ See Churchill [2], p. 35.

Therefore

$$(\rho_1 I-D)L(s) = \rho^{-1}c - \lambda\rho_2 A^{-1}p + q + \rho_3\, s$$

where $\rho_3 = \rho-(\alpha+\delta)$.

b. Solution.

Expressions for $L(s)$, $L(\dot{q})$ in terms of initial values (q, s, λ) have been obtained. System (34) can now be solved for (q, ψ, λ), given (s, z, p), under the assumption of a constant rate of "memory loss", α, and with the utility function as specified in this section. System (34), in this case, is

$$\begin{aligned}
&\text{(a)} \qquad a + Aq + Bs + \psi = \lambda p \,, \\
&\text{(b)} \qquad p'q = z - p'\, L(\dot{q}) \qquad\qquad (35) \\
&\text{(c)} \qquad q + \rho s = \rho_4 L(s) - L(\dot{q}) \,, \qquad \rho_4 = \rho(\rho+\alpha)
\end{aligned}$$

with $L(\dot{q})$ and $L(s)$ as derived above:

$$L(\dot{q}) = Eq + Fs + \lambda Gp + d \,,$$

$$(\rho_1 I-D)L(s) = \rho^{-1}c - \lambda\rho_2 A^{-1}p + q + \rho_3\, s$$

The vector ψ appears only in (35a). Therefore, the solution for (q, λ) can be obtained from the $(n+1)$ equations (35b) and (35c). This is what we are mainly interested in. The equilibrium value of ψ - the utility valuation of additions to consumption capital at the beginning of the plan - can be found by inserting the solution (q, λ) from (35b), (35c) into (35a).

After some manipulation, it is possible to write (35b), (35c) as

$$\begin{aligned}
&\text{(b)} \qquad p'(I+E)q = z - \lambda p'Gp + p'H_1(s) \\
&\text{(c)} \qquad q = -\lambda Kp + H_2(s)
\end{aligned} \tag{35}$$

where

$$-H_1(s) = d+Fs \ ,$$

$$-H_2(s) = (I+E-\rho_4 D_o)^{-1}\left\{(\rho I+F-\rho_3\rho_4 D_o)s + (d-\rho^{-1}\rho_4 D_o c)\right\} ,$$

$$K = \rho_2 \{I + E - \rho_4 D_o\}^{-1} \{\rho_4 D_o + (\mu H)^{-1} E\} A^{-1}$$

and $D_o = (\rho_1 I-D)^{-1}$. Once this system of (n+1) equations in (q, λ) is written in this form, the scalar λ can be easily eliminated. Premultiplying (35c) by the row vector $p'(I+E)$ and then equating (35b) and (35c) we obtain

$$\lambda^* = (p'Jp)^{-1} \{ z + p' H_3(s) \}$$

where

$$J = G - (I + E)K \ ,$$

$$H_3(s) = H_1 - (I+E)\ H_2 .$$

Substituting this value of λ into (35c) we obtain the system of demand equations in the case of quadratic utility with consumption capital,

$$q = -\lambda^* Kp + H_2(s) \ . \tag{36}$$

This system is a "linear" form in prices and consumption capital only. The price coefficients of this form are themselves rational expressions in (z, p, s).

The consumption function that corresponds to (36) is obtained premultiplying by p' :

$$p'q = \nu(z + p'H_3(s)) + p'H_2(s), \tag{37}$$

where ν is the ratio of quadratic forms

$$\nu = -\frac{p'Kp}{p'Jp} .$$

Let $p'q \equiv v$. Writing z in terms of u in (37), and substituting this value of z into the system of commodity demand equations we obtain

$$q = \frac{v - p'H_2(s)}{p'Kp} Kp + H_2(s) \quad ,$$

the system of commodity demand equations with total consumption expenditures, prices and consumption capital as the explanatory variables. Therefore, it is natural to name "EQUS with consumption capital" the system of demand equations and the corresponding consumption function:

$$q = \frac{v - p'H_2(s)}{p'Kp} Kp + H_2(s) \ , \tag{38}$$

$$v = \nu[z + p'H_3(s)] + p'H_2(s). \tag{39}$$

where $\nu = -(p'Jp)^{-1}(p'Kp)$.

Expected inflation can be accounted for by writing $\hat{z}$ instead of z in (39).

REFERENCES

[1] Athans, M., and Falb, P. L., Optimal Control, (New York: McGraw Hill), 1966, xiv + 879 pp.

[2] Churchill, R. V., Operational Mathematics, (New York: McGraw Hill), 1958, ix + 337 pp.

[3] Goldberger, A. S., "Functional Form and Utility: A Review of Consumer Demand Theory", University of Wisconsin, Social Systems Research Institute, Systems Formulation, Methodology and Policy Workshop Paper 6703 (October 1967), pp. 122 (mimeo).

[4] ________, and Gamaletsos, T., "A Cross-Country Comparison of Consumer Expenditure Patterns", European Economic Review, Vol. 1, No. 3 (Spring), 1970, pp. 357-400.

[5] Houthakker, H. S., and Taylor, L. D., Consumer Demand in the United States: Analyses and Projections, second (enlarged) edition (Cambridge, Mass.: Harvard University Press, 1970).

[6] Lluch, C., and Morishima, M., "Demand for Commodities Under Uncertain Expectations" in M. Morishima (ed), Theory of Demand, Real and Monetary (Oxford: U. P.), Ch. V. (forthcoming).

[7] Parks, R. W., "Systems of Demand Equations: An Empirical Comparison of Alternative Functional Forms", Econometrica, Vol. 37, No. 4 (October 1969), pp. 629-50.

[8] Phlips, S., "Dynamic Demand Systems and the Maximum Principle", Institut des Sciences Economics, University of Louvain, 1971 (mimeo).

[9] Pollak, R. A., "Habit Formation and Dynamic Demand Functions", Journal of Political Economy, Vol. 78, No. 4 (July-August), 1970, p. 745-763.

[10] Solari, L., Theorie des choix et fonctions de consommation semi-agregees, Modeles statiques, (Geneve: Droz), 1971.

[11] ________, "Experiences econometriques dans le domaine des fonctions de consommation semi-agregees", Universite de Geneve, Centre d'Econometrie (May 1971),

[12] Stone, R. A., "Linear Expenditure Systems and the British Pattern of Demand", Economic Journal, Vol. 64 (1954), pp. 511-27.

[13] Von Weizsacker, C. C., Notes on Endogenous Change of Tastes, Journal of Economic Theory, Vol. 3., No. 4 (December, 1971), pp. 345-372.

December 22, 1971

PRIVATE, NATIONAL AND INTERNATIONAL RETURNS; AN APPLICATION TO COMMODITY LENDING

L. M. Goreux*

Page

---

* This paper is based on work carried out by the author while at the International Bank for Reconstruction and Development. The author wishes to acknowledge his gratitude to Prof. A. Manne, J. Waelbrocck and B. Balassa who commented on an earlier version of the paper as well as to several economists of the Development Research Center, I. Adelman, A. Condos, P. Davis, and A. Stoutjesdijk. However, the views presented in this paper are solely the responsibility of the author and should not be interpreted as representing the official opinion of the International Bank for Reconstruction and Development.

PRIVATE, NATIONAL AND INTERNATIONAL RETURNS; AN APPLICATION TO INTERNATIONAL COMMODITY LENDING

This paper investigates how some of the welfare concepts, developed for allocating public funds within a country could be extended to the allocation of international funds by an international project lending agency. By analogy with the distinction between private and social returns within a country, a distinction is drawn between national and international returns.

The case of investments affecting commodity prices is used here to illustrate conflicts between private, national and international returns. Since various parties buying and selling the commodity are affected by the price variations induced by the investment, returns of the investment are defined in relation to each affected party. These returns are measured by the impact of the investment on the value of the objective function of the party concerned. The private return to the producer (or group of producers) i is defined as the contribution of the investment to i's profit. The national return to country j is defined as the contribution of the investment to the combined producers-consumers surplus in country j. The international return to an international project lending agency is defined as the contribution of the agency's lending activities to the international welfare function characterizing the objectives of that agency.

By drawing a distinction among the returns of an investment to various parties, the paper shows how income distribution within a country or among countries can be influenced by national or international investments policies. The distinction between the returns to different parties is particularly relevant when the investment decision is not taken by a simple decision agent, but is influenced by several agents, each reacting to the impact of the investment on the value of his own objective function. The paper is divided into three parts. The first deals with the conflicts among various interest groups within a country, the second with the conflicts among trading countries and, the third with the conflicts between national and international returns in the case of international commodity lending.

In the first part, international prices are taken as exogenous to the investment decision made in the country, but domestic prices are allowed to vary between a higher and a lower limits, defined by the import and export prices. The sector is divided into the project area (p) and the area outside the project (op). The national return is subdivided into three components: the consumers' surplus (c), the surplus to producers (p) and the surplus to producers (op).

In the second part, the international commodity price is a variable endogenous to the investment decision model. Each country (or group of countries) tries to maximize its own return, taking the demand and supply curves of its trading partners as exogenous. This restriction is relaxed in the last section where interactions among trading partners are studied in the context of a dynamic game.

The last part differs from the second by the introduction of an international agency lending to the LDC's only. It outlines a model maximizing an international welfare function $\sum_j w_j Y_j$, where Y_j is country j's gain and w_j a weight negatively correlated with j's average per caput income. The indirect benefit (or loss) accruing to country j on account of the agency's lending in countries other than j are included in the country's gain Y_j for various types of investment t. In addition to constraints on availability of international funds, constraints are imposed on the minimum benefits $\overline{Y}_j$, which should accrue to each country on account of the agency's lending activities. With these model specifications, the agency should differentiate its lending rates by countries j and types of investment t. The model solution gives the matrix r_{jt} defining the cut-off rates of national returns by countries j and types of investment t. Short of taking into account interactions between j and t in an optimizing model, two separate vectors r_j and r_t could be defined. Short of defining the vector r_t, the agency could draw a black and white commodity list.

1. Private versus National Return

In drawing a distinction between private and social returns, Little and Mirlees have concentrated their attention in correcting the price distortions caused by protection and by the excess of wages over the opportunity cost of labor [1]. We follow Little and Mirlees and measure the social return of an investment in relation to the price of internationally traded commodities. In this first part of the paper, we restrict our analysis to the case of commodities f for which the import and export prices are exogenous data for country j;(we assume that country j accounts for a small share of the world trade for commodities f). But we depart from Little and Mirlees by simultaneously recognizing the difference between the import and export prices and treating country j's trading pattern as an endogenous variable. If it is not known *ex ante* whether country j will import commodity f, will export it or will be self-sufficient, the marginal utility of commodity f to country j's consumers is an endogenous variable bounded upwards by the import price and downwards by the export price.

If the investment induces a price decline, the national return (defined as the contribution of the investment to the combined producers-consumers surplus) exceeds the return to country j's producers by the gain accruing to country j's consumers. Maximizing the national return thus defined leads to the competitive equilibrium solution; when each producing agent i is a price taker, i's marginal return is identical to the prevailing market price and, consequently, to the marginal utility of commodity f to country j's consumers. But when decision agent i faces a less than infinite price elastic demand for its products, i's marginal return is lower than the marginal utility to consumers defined by the market price; this non-competitive equilibrium solution does not lead, therefore to a Pareto optimum. The difference between social and private returns is analyzed below by considering first the model of a sector producing commodity f and, second, the model of a project accounting for a substantial share of the sectoral output. Let us start with the social return in a sectoral model.

The Sector

Let us consider a one-commodity sector, for example, the sector producing the entire rice output of the country. This sector can sell its rice at a fixed price $\overline{P}_m$ as long as the country imports rice and at a lower price $\overline{P}_x$ if the country exports rice. Let us assume, for example, that the CIF and FOB prices are respectively equal to 100 and 80 and that the average domestic transportation cost is 10 from the producing to the consuming area or from the producing area to the port. In the absence of any tariffs or taxes, the fixed prices $\overline{P}_m$ and $\overline{P}_x$ would be, in this case, respectively equal to 110 and 70. If the value added by the rice sector is a small fraction of the national income, the level of the national price-quantity demand curve for rice can be considered as an exogenous datum for the rice producing sector. Consequently, the demand curve ABCD shown in Figure 1 can be treated as an exogenous datum for the sector.

Let us assume that a clear-cut distinction can be made between sectoral and national resources. All sectoral resources (land, water, local unskilled labor) cannot be used outside of the rice sector. All national resources (fertilizers, fungible capital, etc.) can be bought by the sector at fixed prices in unbounded quantities. The supply curves (S) shown in Figure 1 measure the cost of these national resources. The social utility added by the resources specific to the rice sector is then equal to the area between the demand curve ABCD and the sectoral supply curve. It corresponds on Figure 1 to areas AMK, ABM'K' or ABCM"K", depending on whether the country imports, is self-sufficient or exports.

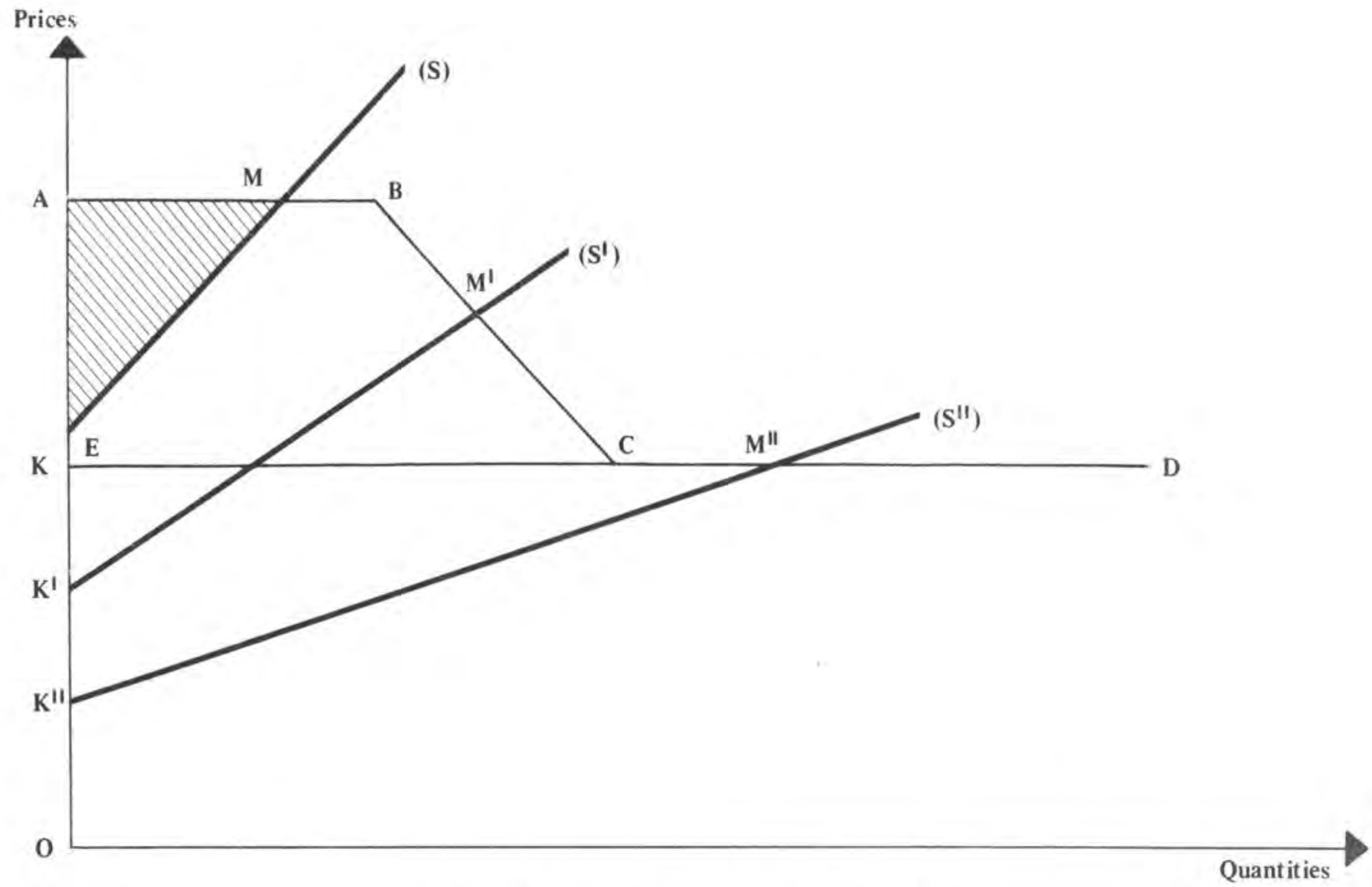


Figure 1: DEMAND CURVE ABCD

World Bank—6303

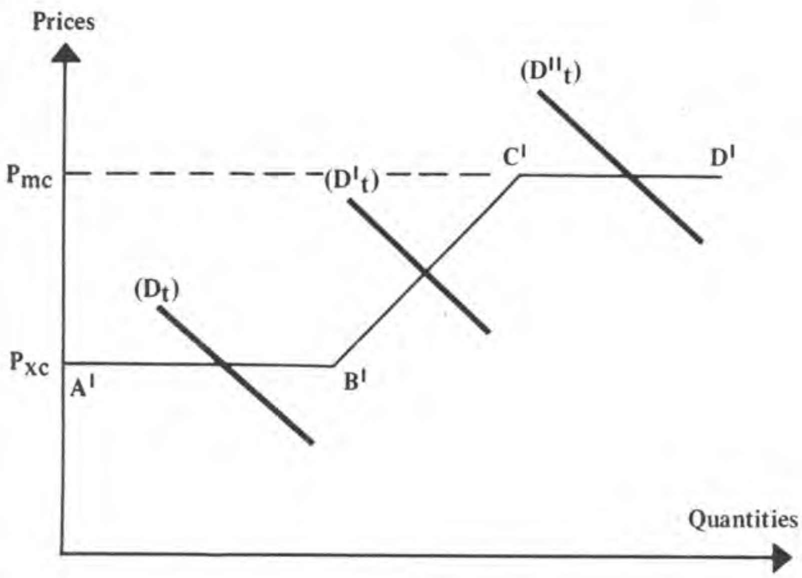


Figure 2a: TEXTILE INDUSTRY, DEMAND FOR COTTON (D_t) AND SUPPLY OF COTTON (A' B' C' D')

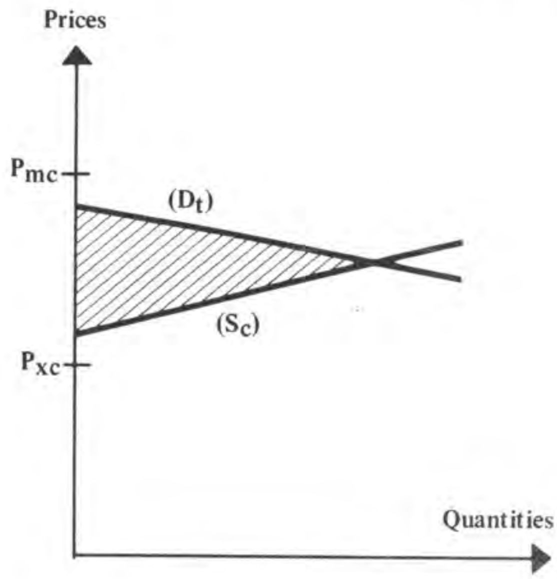


Figure 2b: DEMAND FOR COTTON BY THE TEXTILE INDUSTRY (D_t) AND COTTON DOMESTIC SUPPLY (S_c)

World Bank—6304

Had we known, ex ante, that rice would either be always imported or be always exported, we could have applied Little and Mirlees' method. But if the structure of the trading pattern is not a datum but an unknown, we need to include in the model the demand curves ABCD for the products of the sector.

The sectoral model can be solved without being embedded in an economy-wide model if the sector (a) can be treated as a price-taker for the national resources it uses; (b) accounts for a modest share of the national income and therefore does not have a significant impact on the level of the demand curve for its products, and; (c) produces final goods for consumption or exports. If conditions (a) and (b) are approximately fulfilled, while condition (c) is not, the boundaries of the sector may have to be expanded, as shown in the example below.

Let us assume that cotton can be imported at the price $P_m = 13$ and exported at the price $P_x = 10$. If a domestic textile industry is profitable, even when cotton has to be imported at 13, the demand curve for the cotton produced by the agricultural sector can be represented by the curve ABCD of Figure 1. The quantities AB and EC correspond to the cotton requirements of the domestic textile industry at the prices of 13 and 10 respectively. →

The problem is therefore identical to the one described previously for rice. Alternatively, if cotton production is profitable even if it has to be exported at price $P_x = 10$, the social return of the textile project can be computed by valuing the cotton input according to the curve A'B'C'D' of Figure 2A. The quantities A'B' and E'C' correspond to the levels of domestic cotton production at the prices of 10 and 13, respectively.

If the cost of producing cotton is between 10 and 13, while textile production is profitable only if the price of cotton is lower than 13, it may be profitable to produce simultaneously cotton and textiles while each operation in isolation would have been unprofitable. As shown in Figure 2B, the social optimum consists in producing the quantity OE of cotton and transforming it into textiles. The social gain corresponds to the hatched area of the triangle ABC. The solution is to expand the boundaries of the sector so as to include cotton growing and textile manufacturing in a single model. [7]

The Project Area

Let us now turn to the case of a project-area (p), which accounts for only part of the sectoral production. Let us put aside the case of a sector-wide program (for example, the introduction of improved rice seeds) which is tested in the project area (p), taken as representative of the entire sector. If the model of the project area is designed to provide a representative sample of the sectoral model, the percentage increase of rice production resulting from the application of improved seeds should be the same in the project area (p) and in the entire sector. The elasticity of the demand for rice in the model of the project area (p) should therefore be the same as the elasticity of the demand for rice in the sector. We shall not elaborate on this case and we shall restrict our attention below to the case of an investment which can be implemented in the project area (p), but cannot be duplicated outside of the project area called (op).

For example, there is one single site for an irrigation dam and this site is located in area (p) and not in area (op).

Areas (p) and (op) are each endowed with area-specific resources (land, water, unskilled labor, etc.), which cannot be employed outside. Areas (p) and (op) compete for the production of commodity f ; the total demand for that commodity (produced in p and op) is given by the curve ABCD on the upper part of Figure 3. Areas (p) and (op) are price-takers for all factors other than the area-specific resources. The supply curves (Sp), (Sop) and (S) measure the marginal production costs in (p), (op) and (p) + (op). Only the cost of the national resources bought at fixed prices by areas (p) and (op) are accounted for in these supply curves. The return to the area-specific resources is measured by the surface between the demand and the supply curves.

Without the project, the supply curves are respectively (Sp), (Sop) and (S). The curves (S) and (Sop) are shown on the upper part of the diagram, while the curve (Sp) obtained by taking the difference between the former two is shown for clarity on the lower diagram. The impact of the project is to shift the supply curve in the project area from (Sp) to (S'p) and, as a result, the sectoral supply curve from (S) to (S'). By assumption, the project, which is located in area (p), has no impact on the supply curve (Sop) of area (op).

In (p)'s decision model, the sectoral demand curve ABCD and the supply curve (Sop) for the rest of the sector are given. Consequently, the demand curve Ep Bp Vp Dp for the output originating from project area (p) is also an exogenous datum which can be constructed by difference, as shown in the lower diagram. At the new equilibrium point M'p, the elasticity η_D^p of the demand for the output originating from area p is given[1] by:

$$\eta_D^p = \frac{\eta_D - (1-\alpha_p)\ \eta_{Sop}}{\alpha_p}$$

(1) See equation (15') p. 31.

with α_p = share of area p = HM'/GM'

η_D = elasticity of total demand at point M'

η_{Sop} = elasticity of supply in (op) at point H

This formula shows that the absolute value of the demand elasticity for the project area is always greater than for the sector.[1/] If either the supply elasticity in the rest of the sector η_{Sop}, or the demand elasticity for the sector, η_D is infinitely large, or the share of the project area α_p is infinitesimally small, the demand elasticity for the project area η_D^p is infinitely large and the project area can be treated as a price-taker. In Figure 3, none of these three conditions is fulfilled and the price of commodity f is an endogenous variable in (p)'s decision model. Such a case is not unusual. Often, a large project can be treated as a price-taker for its inputs but not for its outputs.[2/]

In the case of Figure 3, without project the country was importing the quantity MB (=Mp Bp) and the price was OA (=Op Ep). With the project, the country becomes self-sufficient, the price falls to OG (= Op Hp), domestic consumption increases by NM', production in the project area increases by HM' - EM = HL (op's production displaced) + UN (imports displaced) + NM' (increase in domestic consumption). The social gain of the project is

---

1/ Since $0 < \alpha_p < 1$ and since, under normal conditions, $\eta_D < 0$ and $\eta_{S\,op} > 0$, it follows that $|\eta_D^p| > |\eta_D|$.

2/ This generally applies to large irrigation schemes which are to produce fruits and vegetables.

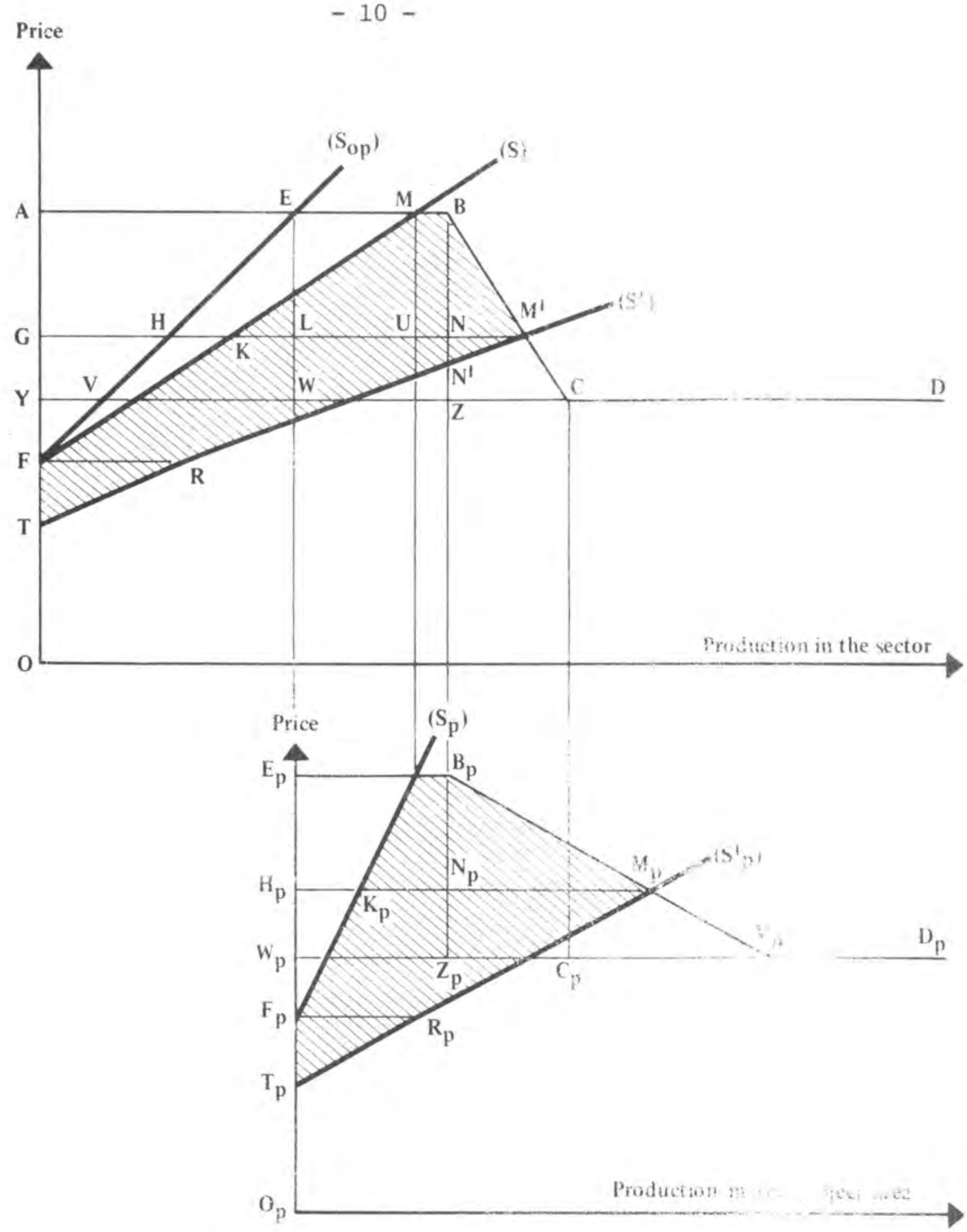


Figure 3: SOCIAL RETURN OF PROJECT (p)

equal to the increment in the return to the area-specific resources in (p) and (op). It is measured on the upper diagram, by the hatched surface MBM'RTF between the demand curve ABCD and the sectoral supply curves S' and S with and without the project. It is also[1/] measured, on the lower diagram, by the hatched surface Mp Bp M'p Rp Tp Fp between the demand curve Ep Bp Vp Dp for the production originating from area (p) and the supply curves S'p and Sp in the project area. The social return of the project can be measured from the model of the project area alone, because the part of the consumers' gain AEHG not accounted for in (p)'s model is exactly compensated by the loss of producers (op) neither accounted either in (p)'s model.

Let us assume, for simplicity, that commodity f is consumed by nationals who do not belong to the sector producing it[2/]. The three groups affected by the price decline are: (1) consumers c who do not own resources specific to the sector; (2) producers p who receive the surplus accruing to the resources specific to the project area (land, local labor, etc.); (3) producers op who receive the surplus accruing to the resources specific to the area op . The social gain represented by the hatched area of the upper diagram is then distributed among these three parties as follows:

1/ Trapezoids MBM'K and Mp Bp M'p Kp have equal areas, since they have the same height and equal basis (AG = Ep Hp, MB = Mp Bp, and KM' = Kp M'p). For the same reason, trapezoids KM'RF and Kp M'p Rp Fp have equal areas (GF = Hp Fp, KM' = Kp M'p and FR = Fp Rp). Finally, triangles FRT and Fp Rp Tp have the same area (FT = Fp Tp and FR = Fp Rp).

2/ This assumption can be relaxed by drawing the demand curves of consumers in (p) and (op). This correction is essential when analysing the agricultural sector of LDC's, since a substantial part of the output is self consumed in the sector.

Consumers c	+ MBM'K		+ EMKH	+ AEHG	= ABM'G
Producers p		+ KM'RTF	- EMKH		
Producers op				- AEHG	
National	+ MBM'K	+ KM'RTF			= MBM'RTF

The project brings to the country a large gain, but this gain is distributed very unequally1/ among the three groups (c), (op) and (p). Consumers (c) make a large gain. Producers (op) make a large loss. In this particular example, producers (p) gain, but producers (p) + (op) together lose.

Programming Model

The obvious advantage of the graphical analysis based on the demand curve for one single commodity is its simplicity. But the area between the demand and the supply curve can be interpreted as a measure of the consumer- producer surplus only at the price of very restrictive assumptions. In particular, if the sector can produce commodities A and B, increasing the production of A is likely to affect the marginal cost of producing B. Consequently, the increment in the area between A's demand and supply curves may be partly offset by a decline in the area between B's demand and supply curves. The increment in the producer-consumer surplus for commodity A alone, therefore, may provide a biased estimate of the net social gain. These difficulties, stressed by I. Little [11], explain why the concept of the consumers-producers surplus which, in the days of A. Marshall, was

1/ This was the result obtained in a two-district agricultural programming model of the Ivory Coast, pp. 57-59 [13]. It could also apply to the impact of large irrigation schemes in California (p) on cotton growers in the South of the United States (op); the result was to accelerate migrations of black workers into the cities.

very much in fashion, fell somewhat out of fashion. However, most of the objections made to the one-commodity analysis disappear, when the commodity demand curves described above are integrated within a large-scale multi-commodity, multi-factor programming model, which can be easily solved on modern computers [12],[13].

Remaining within linear programming techniques, two restrictive assumptions1/ still have to be made; but these restrictions on demand behaviour are relatively weak. The first restriction refers to the substitutability among products. Within a product group (say, cereals) perfect substitutability is permitted within bounds among products (say, between rice and wheat) by allowing any convex combinations of a predominantly rice basket and a predominantly wheat basket. But among product groups (say, between cereals and fruits) substitutability is not permitted. The second restriction is that the amount of utility derived from product (or product group) A has no effect on the utility derived from product (or product group) B. [12]

By allowing the model to import at the price $\overline{Pm}$ and to export at the price $\overline{Px}$, the relevant part of the domestic demand curve is limited to the segment BC. This segment BC, together with the segment B'C' along the marginal revenue curve, are approximated by staircases2/, as shown on Figure 4. Each

---

1/ The income effect on the level of the demand curve can be introduced by iterations between the sectoral and the central models.

2/ It may be noted that approximating the section B'C' of the marginal revenue curve by horizontal steps is equivalent to approximating the price-quantity demand curve by segments of equilateral hyperbolae.

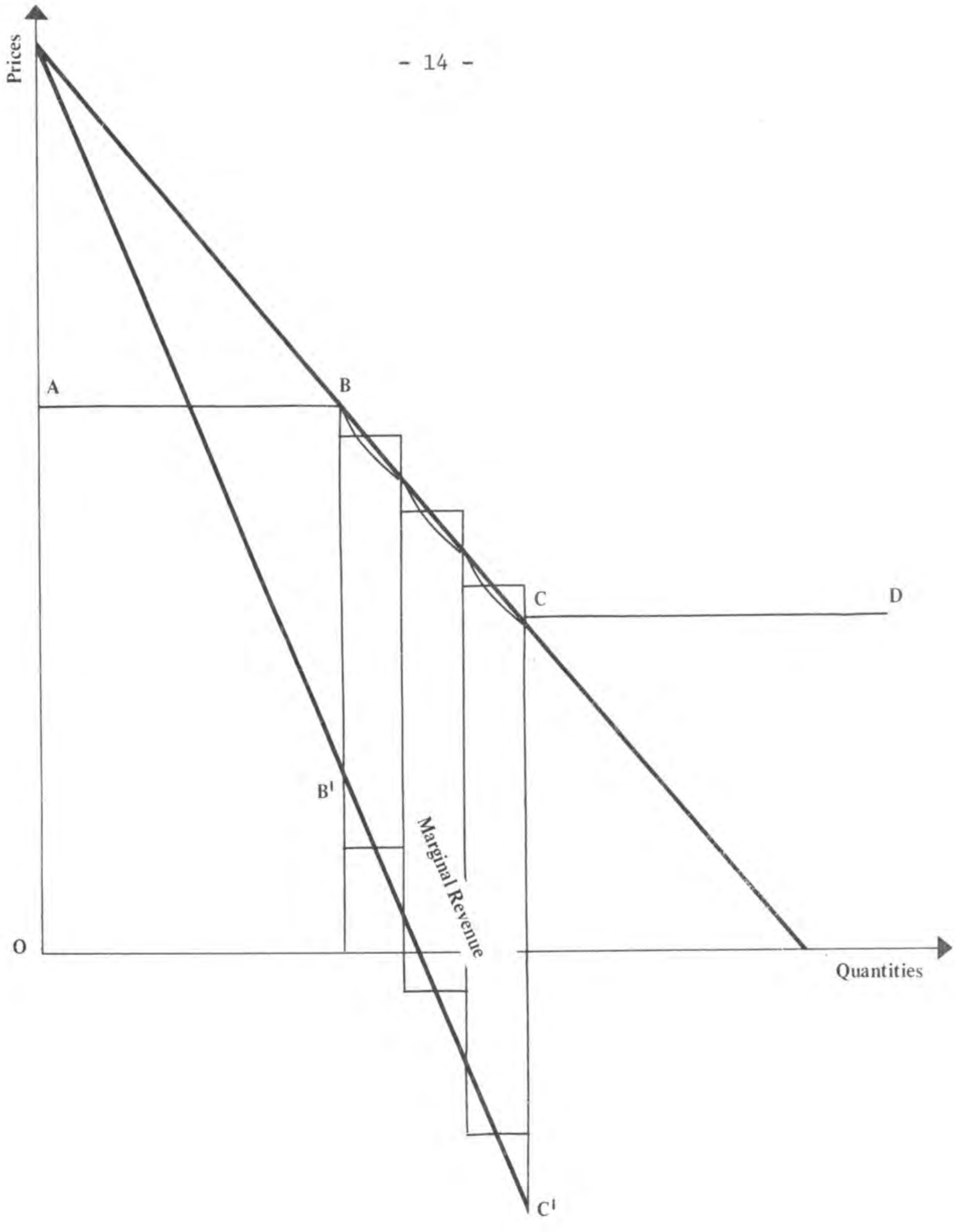


Figure 4: LINEAR APPROXIMATION OF THE PRICE AND MARGINAL REVENUE CURVES

World Bank – 6306

step s (1, ..., n) is bounded in length $Q_s < \overline{Q}_s$. The utility of the additional consumption is $\sum_{s=1}^{s=n} \overline{P}_s Q_s$, the increase in the producers surplus is $\sum_{s=1}^{s=n} \overline{MR}_s Q_s$ - costs. The increase in the consumers surplus is $\sum_{s=1}^{s=n} (\overline{P}_s - \overline{MR}_s) Q_s$. The volume of imports Q_m is obtained by constraining the model to fulfill the minimum requirement $\overline{Q}_o$ (length of segment AB) by either producing or importing. The volume of exports Q_x is given by the length of the last step used along segment CD. The utility provided by the quantity $\overline{Q}_o$ exceeds $\overline{P}_m \overline{Q}_o$ by a constant. Neglecting this constant, the combined producer-consumer surplus can be written:

$$U = \overline{P}_m \overline{Q}_o - \overline{P}_m (\overline{Q}_o - Q_o) + \sum_{s=1}^{s=n} \overline{P}_s Q_s + \overline{P}_x Q_x - C(Q)$$

$$\begin{bmatrix}\text{utility}\\ \text{of mini-}\\ \text{mum re-}\\ \text{quirement}\end{bmatrix} - \begin{bmatrix}\text{cost of}\\ \text{imports}\end{bmatrix} + \begin{bmatrix}\text{utility of}\\ \text{additional}\\ \text{consumption}\end{bmatrix} + \begin{bmatrix}\text{receipt}\\ \text{from}\\ \text{exports}\end{bmatrix} - \begin{bmatrix}\text{cost}\\ \text{of pro-}\\ \text{duction}\end{bmatrix}$$

Considering the segments AB and CD as steps[1] o and n+1, the utility added U and the producer surplus PS can be written:

$$U = \sum_{s=o}^{s=n+1} \overline{P}_s Q_s - C(Q)$$

$$PS = U - \sum_{s=1}^{s=n} (\overline{P}_s - \overline{RM}_s) Q_s$$

with $Q_s \leqslant \overline{Q}_s$ s(0, ..., n)

$$Q_m + Q_o - \overline{Q}_o = 0$$

$$Q_x - Q_{n+1} = 0$$

$$Q - \sum_{s=o}^{s=n+1} Q_s = 0$$

1/ Without loss of accuracy, the number of rows can be reduced by using convex combinations of activities selling the cumulated amounts $\overline{CQ}_s = \sum_{r=o}^{r=s} \overline{Q}_r$ for a utility $\overline{U}_s = \sum_{r=o}^{r=s} \overline{P}_r \overline{Q}_r$. The two rows required are then the convexity constraint $\sum_s \lambda_s = 1$ and the commodity balance $Q - \sum_s \lambda_s \overline{CQ}_s = 0$. The utility U is then given by $U = \sum_s \lambda_s U_s$. [12]

To measure the social return of an investment, a distribution has to be made among three types of commodities:

(a) Commodities f_m, which will be always imported at the price $\overline{P}_{f_m}$

(b) Commodities f_x, which will be always exported at the price $\overline{P}_{f_x}$

(c) Commodities f which may be imported or exported or for which the country may be self-sufficient

The contribution of the investment to the combined producer-consumers' surplus ΔU, to the producers' surplus ΔPS and to the consumers' surplus ΔCS can be written:

$$\Delta U = \sum_{f_m} \overline{P}_{f_m} \Delta Q_{f_m} + \sum_{f_x} \overline{P}_{f_x} \Delta Q_{f_x} + \sum_{f} \sum_{s} \overline{P}_{fs} \Delta Q_{fs} - \text{costs}$$

$$\Delta PS = \Delta U - \sum_{f} \sum_{s} (\overline{P}_{fs} - \overline{RM}_{fs}) \Delta Q_{fs}$$

$$\Delta CS = \Delta U - \Delta PS$$

If the investment affects only the output of commodities f_m and f_x, the social gain ΔU is equal to the gain of the producers ΔPS. If it affects the output of commodities f, conflicts of interest may arise among the various agents involved in the decision-making process.

Let us start with the case of a single decision agent which has a perfect monopoly and faces a demand curve ABCD for its product. If at the production level AB, the marginal production cost exceeds the export price, the cartel's interest is to produce only AB. In relation to the free competition solution, the social loss is represented on Figure 3 by the area of triangle BM'N'. If the demand is price-inelastic and the supply is price-elastic, this loss remains modest, but the change in income distribution can be very substantial. In relation to the free competition solution, the consumer's loss is represented by area ABM'G and the producer's gain by rectangle ABNG minus triangle NM'N'. If, at production level AB, the marginal production cost were lower than the export price, the cartels' interest would be to act as a discriminating monopolist by selling to domestic consumers at price OA and selling abroad at price OY. The programming model optimizing the cartels' return will therefore fulfill step AB first, then skip step BC and go directly to step CD.

Let us now turn to the case of a government who has to make the decision of building or not building an irrigation dam. Clearly, the government (unlike the monopolist) has to include the consumer surplus in measuring the social return (ΔU) to the investment. But the ways in which the government should recover the initial capital cost depends on the distribution of the producer-consumer surplus among the various parties. If most of the social gain goes to consumers in the form of lower food prices, there is an argument for financing most of the dam from the general budget. However, reducing the price of water would contribute to increase the gains of the producers in area (p) whose income level might be satisfactory even without heavy subsidies. It would not help producers in area (op) who are those losing from the scheme and who, before the scheme, might have been poorer[1/] than producers in area (p). In selecting public investments, the government should therefore give attention to the income distribution effects of these investments which depend on labor mobility between areas and sectors.

1/ This could be illustrated by comparing in Mexico the rich farmers in the North-West irrigated perimeters with the poor farmers of the high plateaus.

We have contrasted the cartel maximizing the income accruing to the cartel's members and the government agency maximizing the national consumer-producer surplus. In practice, there are few watertight cartels and few governments whose sole objective is to maximize the sum of consumers' and producers' surplus. There are often pressure groups, lobbies and considerations of income distribution which affect the nature of the government objective function or introduce additional constraints in the government decision model. It is therefore probably more important to analyze the trade-offs among various objectives than to choose the optimal investment pattern on the basis of a single objective.

2. Trading Partners Maximizing their National Returns

In the previous part, we assumed that an increase in the production of a commodity did not affect the world price level. The analysis was therefore limited to the impact of domestic price variations on various groups within the country. In this part, we assume that an increase in the production of commodity k in country j has an impact on the world price of that commodity and, consequently, on the various countries trading that commodity with j.

Within a country, we had previously drawn a distinction among three interest groups: the producers (p) in the project area, the producers (op) outside this area, and the consumers (c). Now, we shall draw a distinction among the exporting country j which has to make an investment decision, the other exporting countries (o j) and the importing countries. Exporting country (j) replaces, in this part of the paper, the group of producers (p) considered in the previous part. The problem remains basically the same; the solution is different because the roles of the decision agents (p) and (j) are quite different. In the previous part, the main decision agents were the individual farmers and the government. The group of farmers in the project area (p) was generally a loose unit which could influence government decisions only through lobbying. In this part, country j is a major decision agent who can raise export taxes and import duties. Previously, agent (p) was able to influence government decisions, as the weak baron was able to influence the will of an absolute monarch. Now, agent (j) is the medieval baron who does not obey any king.

The difference in the decision making process is reflected in the formulation of the model. Before, social welfare was generally used as the objective function, while the return to group (p) was introduced only as an accounting row or a constraint. Now the return to country j is the objective function.

Due to the existence of several optimizing agents, the analysis is conducted in two stages. In the first section, we optimize country j's decision, taking the demand and supply curves of countries other than j as exogenous to country j's decision. In the second section, we analyze the interactions among the decisions of the various trading partners within the context of game theory.

2.1 Comparative Statics

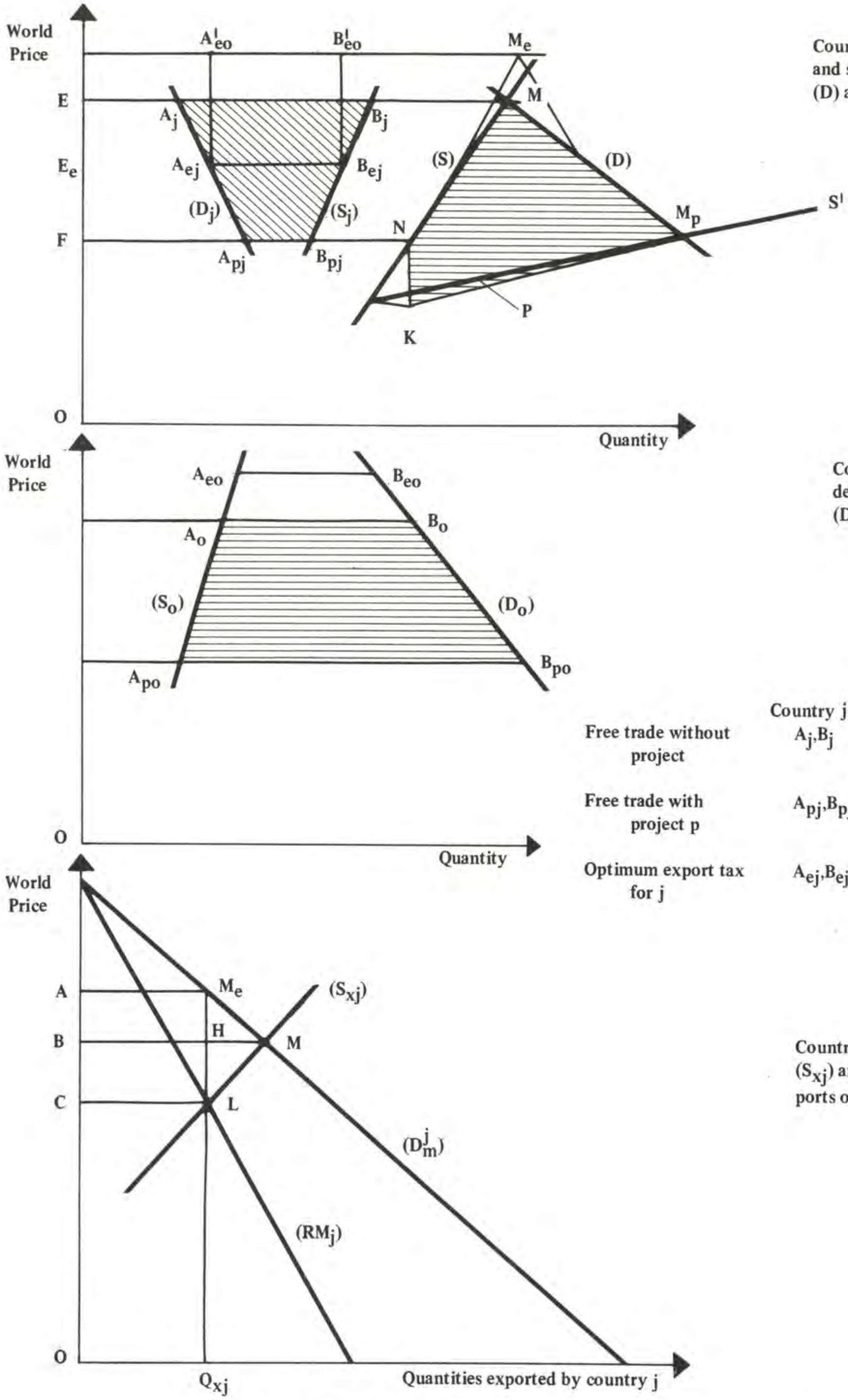
Before considering the pricing of commodities in a national programming model, despite all the restrictive assumptions required, let us start with the one-commodity graphical analysis.

2.1.1 Graphical Analysis

In the absence of any tax or tariff, the equilibrium point M (Q, P) is at the intersection of the world demand and supply curve (D) and (S) shown in the upper diagram of Figure 5. The quantity A_jB_j (upper diagram) exported by country j is equal to the quantity A_oB_o (middle diagram) imported by countries o. Let us now depart from this equilibrium by launching a project in island (p) which had never consumed nor produced the commodity before. The appearance of this new exporter on the world market has the effect of shifting the world supply curve from (S) to(S') and of reducing the world price from OE to OF. The impact of this shift on the combined producer/consumer surplus of island (p) and of countries i and o is as follows:

island (p)	$+ NKM_p$			$= NM_pK$
country j		$- A_jB_jB_{pj}A_{pj}$		$= - A_jB_jB_{pj}A_{pj}$
countries o		$+ A_jB_jB_{pj}A_{pj}$	$+ NMM_p$	$= A_oB_oB_{po}A_{po}$
world	$+ NKM_p$		$+ NMM_p$	$= NMM_pK$

If "world welfare" is defined as the sum of the national consumer/producer surpluses, project (p) has increased world welfare. But the gain in "world welfare' has not been distributed equally between the three parties j, o and p. Country j has suffered a loss. Countries o have captured j's loss and have, in addition, shared the net world gain together with island (p). Let us now imagine that island (p) belongs to country j. By equalizing the marginal production cost in the project area (p) with the world price which would prevail with the project (net of transportation costs),



Country j and world demand and supply curves (D_j), (S_j), (D) and (S)

Countries o (= world −j) demand and supply curves (D_o), (S_o)

	Country j	Countries o	World
Free trade without project	A_j, B_j	A_o, B_o	M
Free trade with project p	A_{pj}, B_{pj}	A_{po}, B_{po}	M_p
Optimum export tax for j	A_{ej}, B_{ej}	A_{eo}, B_{eo}	M_e

Country j export supply curve (S_{xj}) and demand curve for exports originating from j (D_m^j)

Figure 5: IMPACT OF PROJECT (p) UNDER FREE TRADE AND OF THE OPTIMUM EXPORT TAX FOR COUNTRY j

World Bank–6307(R)

country j would lose if, as in this case, area $A_j\ B_j\ B_{pj}\ A_{pj}$ exceeds the area KNM_p. Clearly, country j's interest is not to implement project (p).

To compare the free trade solution and the optimum solution for country j, it is convenient to shift to the lower diagram of Figure 5. (S_{xj}) is the marginal cost curve for country j's exports, (D^j_m) is the demand curve for exports originating from j and (RM_j) is the marginal return curve from j exports. The free trade equilibrium point M is at the intersection of (S_{xj}) and (D^j_m). The optimum equilibrium for country j is at point L, where (S_{xj}) and (RM_j) intersect. By establishing the export tax LM_e, country j exports a smaller quantity AM_e at a higher world price OA and consumes a larger quantity $E_e\ A_{ej}$ at a lower internal price OE_e. In country j, both the consumers and the government gain from the export tax, while the producers lose. Country j as a whole gains because the government could redistribute the profit of the tax so as to make both consumers and producers of country j better off with than without the export tax. The impact of the tax on the various trading countries can be summarized as follows:

Country j	$+HBAM_e$	$-HML$			
Countries o (rest of the world):	$-HBAM_e$		$-HMM_e$	=	$-MBAM_e$
World:		$-HML$	$-HMM_e$	=	$-MLM_e$

Country j's export tax results in a second best solution from the world point of view. With free trade, the world's combined producer/consumer surplus would have been higher; consequently the winning countries could have "bribed" the losing ones so that all countries, including country j, could have been better off with free trade. But, country j is better off with an export tax than without an export tax and without a bribe.

2.1.2 National Programming Model

In the programming model, we have considered earlier the case of commodities f for which country j was a price-taker on the world market. Let us now turn to the case of commodity k the international price of which is endogenous as illustrated in Figure 5. This commodity is produced for domestic consumption and for exports. In country j's social objective function, domestic consumption has to be valued according to its utility by the area under the demand curve D_j on the upper diagram. But, the return from exports has to be valued not according to its utility for the importers (area[1/] under the demand curve D_m^j on the lower diagram) but by its return to country j which is equal to the area under curve RM_j in the lower diagram.

The domestic demand curve D_j and the marginal revenue curve from exports RM_j have to be approximated by two staircases, the height $\overline{P}_{k_d s}$ and $\overline{RM}_{k_x s}$ of each step being the same on both staircases. If Q_k refers to domestic production, Q_{k_d} to domestic consumption, Q_{k_x} to export, s being a subscript characterizing the steps along the two staircases, the commodity balance can be written:

$$Q_k - \sum_s Q_{k_d s} - \sum_s Q_{k_x s} = 0$$

$$\text{with } Q_{k_d s} \leq \overline{Q}_{k_d s}$$

$$Q_{k_x s} \leq \overline{Q}_{k_x s}$$

and the entries in the social objective function:

$$\sum_{k_d} \sum_s \overline{P}_{k_d s} Q_{k_d s} + \sum_{k_x} \sum_s \overline{MR}_{k_x s} Q_{k_x s} - \text{Costs}$$

The optimal solution will then be to expand production up to the point where the marginal production cost is equal to both the marginal

---

1/ This would apply to the discriminatory monopolist.

return from exports and the utility from domestic consumption. The level of the optimum export tax will be the vertical distance (LM_e) between the last step used on (RM_j) and the point on curve (D_m^j) with the same abscissa.

By analogy, if country j were a major importer affecting the import price P_{k_m} of commodity k_m, the marginal1/ import cost curve (MC_j) and the domestic demand curve (D) should be approximated by staircases. The commodity balances and the entries in the social objective function could then be written:

$$Q_k + \sum_s Q_{k_m s} - \sum_s Q_{k_d s} = 0$$

$$\sum_{k_d} \sum_s \overline{P}_{k_d s}\, Q_{k_d s} - \sum_{k_m} \sum_s \overline{MC}_{k_m s} \cdot Q_{k_m s} - \text{Costs}$$

The model would equalize the marginal cost of production to both the marginal cost of imports and the utility of domestic consumption. The level of the optimum import tax would then correspond to the vertical distance between the last step used in the marginal import cost curve and the import supply curve.

2.1.3 Multi- Country Model

Let us now consider an indivisible project (p) in a one-commodity n-country model. The problem is to measure the impact of project (p) on each country and to define under which conditions country j (or group of countries g) is better off with than without the project.

Assumptions

(1) The project producing commodity k has no impact on the price of commodities other than k. The international prices of commodities other than k are used, therefore, as numeraire for measuring the benefits of countries j (or groups of countries g) on account of the project p.

1/ Assuming the country is not a discriminatory monopolist.

(2) Country j pays for its imports or receives for its exports of commodity k price $P_j = P_x + T_j$, where P_x is the world export reference price and T_j is a country-specific transportation cost differential. The project p induces a variation in the world reference price from P_x to $P_x + \Delta P_x$ and in the country-specific price from P_j to $P_j + \Delta P_x$; the country-specific margin T_j therefore remains unaffected by the price change ΔP_x.

(3) Import demand and export supply curves (M_j) and (X_j) are defined for each country j in relation to the world reference price P_x and the levels of these curves is not[1/] affected by ΔP_x.

(4) $\gamma_{xj}\ P_{xj}$ measures the opportunity cost of the resources released by reducing country j's exports by one marginal unit. Similarly, $\gamma_{mj}\ P_{mj}$ measures the opportunity cost of country j's marginal unit of imports. The coefficients γ_{xj} and γ_{mj} would be equal to unity with neutral effective protection, full employment and perfect mobility of country j's resources. In practice, the coefficient γ_{xj} is substantially lower than unity for tropical export crops.

(5) The marginal opportunity costs $\gamma_{xj}\ P_{xj}$, $\gamma_{mj}\ P_{mj}$ and the average opportunity cost C_p per unit of project output are measured in relation to international prices using the Little and Mirlees' method or the programming approach described earlier.

(6) Within the margin of price variation from P_x to $P_x + \Delta P_x$ induced by the project, the export supply and import demand curves can be approximated linearly. Similarly, the marginal opportunity cost can be approximated linearly from $\gamma_j P_j$ to $\gamma_j\ (P_j + \Delta P_x)$.

1/ This assumption will be relaxed in section 2.2.

(7) The net gain (+) or loss (-) of group g is the algebraic unweighted[1/] sum of the individual gains or losses of every country j belonging to g.

After having reviewed the assumptions, the reader may skip the algebra and go directly to the implications on page 35.

The first subscript x or m characterizes exports or imports. The absence of the first subscript either indicates that the formula applies whether the country imports or exports, or indicates summation over gross imports (-) and gross exports (+). The second subscript p, j or g characterizes the project p, the country j or the group of countries g. The absence of the second subscript refers to the world as a whole.

The main symbols are summarized below:

Q_{xj} = volume of j's gross exports (+)

Q_{mj} = volume of j's gross imports (-)

$Q_{xg} = \sum_{j \in g} Q_{xj}$ = volume of g's gross exports (+)

$Q_x = \sum_j Q_{xj}$ = volume of world gross exports (+)

$Q_m = -Q_x$ = volume of world gross imports (-)

$\alpha_{xj} = Q_{xj}/Q_x$ = share of j's gross exports in relation to world exports (+)

$\alpha_{mj} = Q_{mj}/Q_x$ = share of j's gross imports in relation to world exports (-)

$\alpha_{xg} = Q_{xg}/Q_x$ = share of g's gross exports in relation to world exports (+)

$\alpha_{mg} = Q_{mg}/Q_x$ = share of g's gross imports in relation to world exports (-)

$\alpha_g = \alpha_{xg} + \alpha_{mg}$ = share of g's net exports (+) or net imports (-) in relation to world exports (+)

$P_{xj} = P_x + T_j$ = Price j receives for its exports

$P_{mj} = P_x + T_j$ = Price j pays for its imports

1/ This assumption will be relaxed in sections 3.1.3 and 3.2.

P_x = World reference export price

$\gamma_{xj}\ P_{xj}$ = Opportunity cost of the resources released by reducing j's exports by one marginal unit

$\gamma_{mj}\ P_{mj}$ = Opportunity cost of the marginal unit of imports

$\Delta P_x = \Delta P_j = \varepsilon\ P_x$ = Price variation induced by the project

η_j = Elasticity of quantities imported or exported by country j in relation to variations of the world reference price P_x

η_m = Elasticity of the world import demand in relation to P_x (-)

η_x = Elasticity of the world export supply excluding the project in relation to P_x (+)

$\eta = -\ \eta_m + \eta_x$ = (+)

$\alpha_p\ Q_x$ = Volume of exports generated by the project

$V_p = \alpha_p Q_x\ (P_x + \Delta P_x)$ = Exports generated by the project valued at the prevailing world reference price

C_p = Average opportunity cost per unit of the project output

π_p = Direct gain (+) or loss (-) attached to the project alone

π_j = Indirect gain (+) or loss (-) incurred by country j on account of the price variation induced by the project

All quantities Q and shares α are counted positively when they refer to gross or net exports, and negatively when they refer to gross or net imports. ΔP_x is counted negatively for price declines and positively for price increase (eradication schemes). π always stands for gain. If the computed value of π is positive, it has to be interpreted as a net gain. If it is negative, it has to be interpreted as a net loss. With these sign conventions, the formulae are valid whether the project induces a price decline ($\alpha_p > 0$) or a price increase ($\alpha_p < 0$), whether the group g has a net export surplus ($\alpha_g > 0$) or a net import surplus ($\alpha_g < 0$), whether the opportunity cost coefficients γ are smaller or larger than unity. Figure 6 is used only for illustrative purposes.

Project p

Without the project, the equilibrium point corresponds on Figure 6 to the point (N) where the world import demand and export supply curves (M) and (X) intersect. The quantity of world exports is Q_x and the world export reference price is P_x. The project (p) generates the quantum of exportable supplies $\vec{LK} = \alpha_p Q_x$ and induces the fall $\vec{NH}$ in the world reference price from P_x to $P_x + \Delta P_x$. The price decline ($\Delta P_x < 0$) results in a reduction $|\Delta Q_x|$ in the world exports originating from outside the project and an increase $|\Delta Q_m|$ in world imports. With the sign conventions used both ΔQ_x and ΔQ_m are negative. ΔQ_x is a reduction in the volume of exports counted as + . ΔQ_m is an increase in the absolute value of imports counted as - .

Without the project, the world trade balance was:

(1) $Q_x + Q_m = 0$

The impact of the project on this balance is:

(2) $\alpha_p Q_x + \Delta Q_x + \Delta Q_m = 0$

$(\vec{LK} + \vec{HL} + \vec{KH} = 0)$

where ΔQ_x and ΔQ_m are defined by:

(3) $\frac{\Delta Q_x}{Q_x} = \eta_x \frac{\Delta P_x}{P_x}$ (4) $\frac{\Delta Q_m}{Q_m} = \eta_m \frac{\Delta P_x}{P_x}$

By combining (1), (2), (3) and (4), the relative price decline ε can be written:

$$(4) \quad \varepsilon = \frac{\Delta P_x}{P_x} = \frac{-\alpha_p}{\eta_x - \eta_m} = \frac{-\alpha_p}{\eta}$$

The size of the project can therefore be characterized either by its share α_p of world exports before the project or by the relative price decline ε induced by the project. If the project output $\alpha_p Q_x$ is sold at the world reference price P_x, by using equation (4), the export value generated V_p is:

$$(5) \quad V_p = \alpha_p Q_x (P_x + \Delta P_x) = - \varepsilon (1+ \varepsilon) \eta P_x Q_x$$

If the project output is exported at the price $P_i = P_x + T_i$, where T_i accounts for transportation cost differential, the export value $V_p^{(i)}$ of the project output is:

$$(6) \quad V_p^{(i)} = \alpha_p Q_x (P_i + \Delta P_i) = \alpha_p Q_x (P_x + \Delta P_x + T_i) = V_p + \alpha_p Q_x T_i$$

If $C_p^{(i)}$ is the average opportunity cost per unit of the project output (measured in terms of internationally tradeable commodities), the direct profit $\pi_p^{(i)}$ of the project (area LKJ' = area LKRJ on Fig. 6) is:

$$(7) \quad \pi_p^{(i)} = V_p^{(i)} - \alpha_p Q_x C_p^{(i)} = V_p \frac{P_i + \Delta P_i - C_p^{(i)}}{P_x + \Delta P_x}$$

<u>Country j where the Project p is not Located</u>

j's gain resulting from the impact of the price decline $\Delta P_x = \Delta P_j$ on j's export earnings (or import bill) from (or for) commodity k is measured by:

$$(8) \quad (P_j + \Delta P_j)(Q_j + \Delta Q_j) - P_j Q_j = Q_j \Delta P_x + (P_j + \Delta P_x) \Delta Q_j$$

To this first effect has to be added the saving resulting from the opportunity cost of the resources released by a reduction in j's exports or the gain from j's additional imports. If this marginal gain is measured (in terms of internationally tradeable commodities) by $\gamma_j P_j$, when the price declines from P_x to $P_x + \Delta P_x$, country j's gain is:

$$(9) \quad -\gamma_j \int_{Q_j}^{Q_j + \Delta Q_j} P_j \ (Q_j) \ dQ_j = - \gamma_j \frac{P_j + \Delta P_j + P_j}{2} \Delta Q_j$$

The minus sign is due to the fact that a price decline[1] ($\Delta P_x < 0$) induces a positive gain and implies $\Delta Q_j < 0$. In the case of Figure 6, the gain from the resources released corresponds to the trapezoid EGIF with $\vec{FI} = \gamma_{xj} P_j \ , \ \vec{EG} = \gamma_{xj}(P_j + \Delta P_j), \ \vec{FE} = \Delta Q_{xj}$

By adding (8) and (9), j's gain π_j can be written:

$$(10) \quad \pi_j = Q_j \ \Delta P_x + \left[(1-\gamma_j)P_j + \Delta P_x - \gamma_j \frac{\Delta P_x}{2}\right] \Delta Q_j$$

Noting that:

$$\frac{\Delta Q_j}{Q_j} = \eta_j \frac{\Delta P_x}{P_x}$$

Replacing ΔP_x by εP_x and introducing the share coefficients α and V_p from (5). $P_x \ \Delta Q_j = \alpha_j \ \eta_j \ \varepsilon \ P_x \ Q_x = - \frac{V_p}{1+\varepsilon} \frac{\eta_j}{\eta} \alpha_j$

$$Q_j \ \Delta P_x = \alpha_j \ \varepsilon \ P_x \ Q_x = - \frac{V_p}{1+\varepsilon} \frac{\alpha_j}{\eta}$$

π_j can be rewritten in relation to V_p by defining a coefficient β_j

$$(11) \quad \pi_j = - \beta_j V_p = - V_p \frac{\alpha_j}{\eta(1+\varepsilon)} \left[1 + \eta_j \left[(1-\gamma_j)\frac{P_j}{P_x} + \varepsilon \ (1 - \frac{\gamma_j}{2})\right]\right]$$

Country j would be better off with than without the project provided the sum of the direct project gain $\pi_p^{(i)}$ and of the indirect impact on country j π_j is positive. Replacing the price P_i by P_j in (7) and

---

[1] On the opposite $\Delta P_x > 0$ implies $\Delta Q_j > 0$ and induces a loss.

combining (7) with (11) the condition $\pi_p^{(j)} + \pi_j > 0$ can be written:

$$(12) \quad C_p^{(j)} < P_j + \Delta P_j - \beta_j (P_x + \Delta P_x)$$

where P_j refers to the import or export price, depending on whether j imports or exports and β_j is a coefficient defined by equation (11). When the relative price decline tends towards zero, the value of this coefficient tends towards the limit β_j^*:

$$(13) \quad \beta_j^* = \frac{\alpha_j}{\eta} \left[1 + \eta_j (1 - \gamma_j) \frac{P_j}{P_x}\right]$$

Equations (14) and (15) show the values of β_j^* for two values of γ_j. For $\gamma_j = 1$, the opportunity cost of the marginal unit traded is identical to the price paid (or received) for it. For $\gamma_j (P_j + \Delta P_j) = C_p^{(j)}$, the saving made by reducing the output of the established producers is identical to the production cost of the new producers; production of commodity v is therefore optimally allocated within country j between the project area and the rest of the country.

$$(14) \quad \beta_j^* = \frac{\alpha_j}{\eta} \quad \text{for} \quad \gamma_j = 1$$

$$(15) \quad \beta_j^* = \frac{\alpha_j}{\eta - \alpha_j \eta_j} \text{for } \gamma_j (P_j + \Delta P_j) = C_p^{(j)}$$

In the case of an exporting country ($\alpha_j > 0$), calling η_{xoj} the elasticity of exports originating from countries other than j and noting that $\eta = - \eta_m + \eta_x = -\eta_x + \alpha_j \eta_{xj} + (1 - \alpha_j) \eta_{xoj}$, (15) can be rewritten in relation to the elasticities in countries other than j:

$$(15') \quad \beta_{xj}^* = \frac{\alpha_j}{-\eta_m + (1-\alpha_j)\eta_{xoj}} \quad \text{for } \gamma_j (P_j + \Delta P_j) = C_p$$

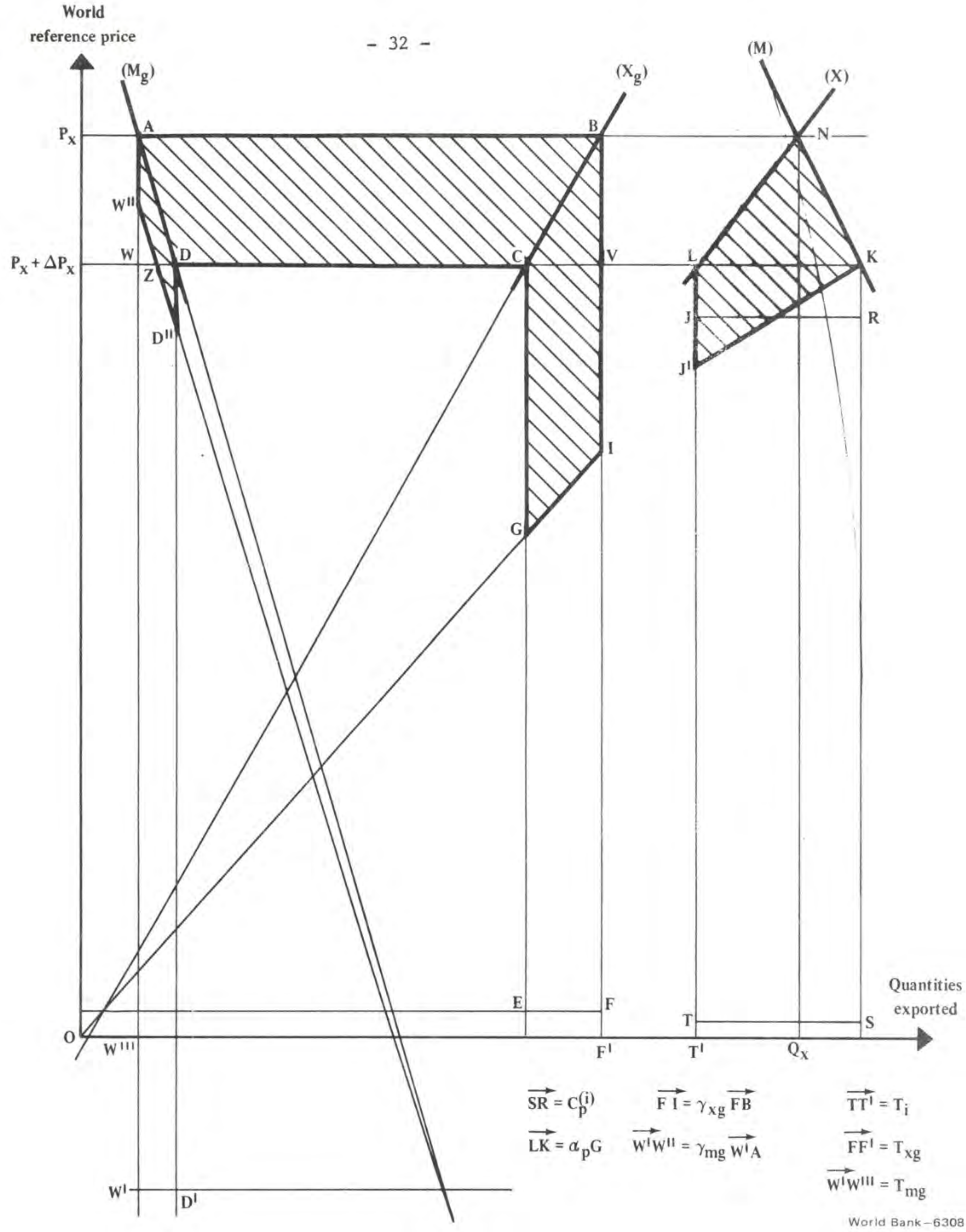


Figure 6: IMPACT OF PROJECT p ON GROUP g ($\alpha_g > 0$)

Group of Countries g

The indirect return π_j to country j is measured in (11) in terms of the international prices of commodities other than k. Assuming that one dollar gain or loss accruing to any country j belonging to g has the same value, the indirect return π_g to group g is defined by:

$$(16) \quad \pi_g = \sum_{j \in g} \pi_j$$

Countries j belonging to g are stratified between exporters xj and importers mj. Average coefficients α_g, η_g, γ_g and P_g are defined for each stratum. For the export stratum, the definitions are given in (17):

$$(17) \quad \alpha_{xg} = \sum_{xj \in g} \alpha_{xj} \qquad \eta_{xg} = \frac{\sum_{xj \in g} \alpha_{xj} \eta_{xj}}{\alpha_{xg}}$$

$$\gamma_{xg} = \frac{\sum_{xj \in g} \alpha_{xj} \eta_{xj} \gamma_{xj}}{\alpha_{xg} \eta_{xg}} \qquad P_{xg} = \frac{\sum_{xj \in g} \alpha_{xj} \eta_{xj} \gamma_{xj} P_{xj}}{\alpha_{xg} \eta_{xg} \gamma_{xg}}$$

For the import stratum, the coefficients can be obtained simply by replacing subscript x by subscript m. The share of net trade α_g is defined by $\alpha_g = \alpha_{xg} + \alpha_{mg}$.

Replacing in (16) π_j by its value given in (11) and using the definitions given in (17), π_g (which corresponds to the hatched area W"ABIGCDD" on Figure 6) can be written:

$$(18) \quad \pi_g = -\beta_g V_p = -\frac{V_p}{1+\varepsilon}\left(\frac{\alpha_g}{\eta} + \alpha_{xg}\frac{\eta_{xg}}{\eta}\left[(1-\gamma_{xg})\frac{P_{xg}}{P_x} + \varepsilon\left(1-\frac{\gamma_{xg}}{2}\right)\right] + \alpha_{mg}\frac{\eta_{mg}}{\eta}\left[(1-\gamma_{mg})\frac{P_{mg}}{P_x} + \varepsilon\left(1-\frac{\gamma_{mg}}{2}\right)\right]\right)$$

$$-\left|\begin{matrix}\text{area}\\ \text{W''ABIGCDD''}\end{matrix}\right| = -\left|\begin{matrix}\text{area}\\ \text{ABVW}\end{matrix}\right| - \left|\text{area CVFE}\right| + \left|\text{area GIFE}\right| - \left|\text{area DWW'D'}\right| + \left|\text{area D''W''W'D'}\right|$$

$$= -\left|\begin{matrix}\text{area}\\ \text{ABVW}\end{matrix}\right| - \left|\text{area CVIG}\right| - \left|\text{area DZD''}\right| + \left|\text{area ZWW''}\right|$$

By combining equations (18) and (11), it follows that group g is better off with than without the project($\pi_g + \pi_p^j > 0$) if:

$$(19) \quad C_p^{(j)} < P_j + \Delta P_j - \beta_g \ (P_x + \Delta P_x)$$

P_j is the specific export or import price for country j where the project is located and β_g is the coefficient defined in (18).

For particular values of γ the values of π_g and β_g can be simplified. Thus, for $\gamma_{xg} = \gamma_{mg} = 1$, π_g (which corresponds to area ABCD in figure 6) takes the form (20):

$$(20) \quad \pi_g = - \frac{V_p}{1+\varepsilon} \left[\frac{\alpha_g}{\eta} + \frac{\varepsilon}{2} \left(\alpha_{xg} \frac{\eta_{xg}}{\eta} + \alpha_{mg} \frac{\eta_{mg}}{\eta} \right) \right]$$

$$- \left| \begin{matrix} \text{area} \\ \text{ABCD} \end{matrix} \right| = - \left| \begin{matrix} \text{area} \\ \text{ABVW} \end{matrix} \right| + \left| \begin{matrix} \text{area} \\ \text{CVB} \end{matrix} \right| + \left| \begin{matrix} \text{area} \\ \text{DWA} \end{matrix} \right|$$

With $\gamma_{xg} = \gamma_{mg} = 0$, π_g measures the loss in net export earnings.

$$(21) \quad \pi_g = - \frac{V_p}{1+\varepsilon} \left(\frac{\alpha_g}{\eta} + \alpha_{xg} \frac{\eta_{xg}}{\eta} \frac{P_{xg}+\Delta P_{xg}}{P_x} + \alpha_{mg} \frac{\eta_{mg}}{\eta} \frac{P_{mg}+\Delta P_{mg}}{P_x} \right)$$

$$- \left| \begin{matrix} \text{area} \\ \text{W'ABFECDD'} \end{matrix} \right| = - \left| \text{area ABVW} \right| - \left| \text{area CVFE} \right| - \left| \text{area DWW'D'} \right|$$

When ε tends towards zero, the coefficient β tends towards β^*

$$(22) \quad \beta_g^* = \frac{\alpha_g}{\eta} + \alpha_{xg} \frac{\eta_{xg}}{\eta} (1-\gamma_{xg}) \frac{P_{xg}}{P_x} + \alpha_{mg} \frac{\eta_{mg}}{\eta} (1-\gamma_{mg}) \frac{P_{mg}}{P_x}$$

which can be written:

$$(23) \quad \beta_g^* = \frac{\alpha_g}{\eta} \qquad \text{for } \gamma_{xg} = \gamma_{mg} = 1$$

If group g is extended to the entire world $\alpha_g = 0$, $\alpha_{xg} = 1$ and $\alpha_{mg} = -1$. It follows that in equation (23) β_g^* becomes zero and that equation (19) becomes $C_p^{(j)} < P_j + \Delta P_j$. We end up with the well known result of the free trade model: the production cost in the marginal project should equate the prevailing world price after adjustment for transportation costs ($P_j - P_x = T_j$).

Implications

The difference between the gain π_j accruing to country j implementing project p and the gain π_g of the group g to which country j belongs is $\pi_j - \pi_g = (\beta_g - \beta_j)\, V_p$. If country j is a marginal exporter, this difference becomes $\beta_g V_p$, where V_p is the value of the gross output of the project. For a cartel or an international project lending agency representing the interests of group g, it is important to take into account the difference $\pi_j - \pi_g$ and therefore to compute β_j and β_g.

The computation of β_j and, in particular, that of β_g from (28) may appear difficult. However, in most practical cases, the output of the project is small in relation to the volume of world exports. Consequently, the relative price decline (ε) induced by the project is also small and β_g^* (or β_j^*) provides a satisfactory[1/] approximation for β_g (or β_j). Thus, for group g, the criterion for project selection given in equation (19) can be replaced by:

$$(24) \quad C_p^{(j)} < P_j - \beta_g^* \, P_x = MR_g$$

where the right hand side measures the marginal return MR_g of a unit of production to group g. The coefficient β_g^* is equal, therefore, to the difference between the price P_j at which the commodity can be exported from (or imported into) country j and the marginal return to group g (MR_g) divided by the world reference price (P_x):

$$(25) \quad \beta_g^* = \frac{P_j - MR_g}{P_x}$$

1/ The reader may wish to calculate from formula (18) the size of the projects α_p above which $|\beta_g - \beta_g^*|$ exceeds the permissible margin of error.

To apply the criterion (24), the relative price decline (ε) induced by the project does not even need to be computed, but the coefficient β_g^* has to be computed. The reason is that, when ε tends towards zero, $P_j - MR_g$ tends towards the non-zero limit $\beta_g^* P_x$.

Under the perfect market assumption ($\gamma_{xg} = \gamma_{mg} = 1$), β_g^* can be easily estimated from (23) $\beta_g^* = \dfrac{\alpha_g}{-\eta_m + \eta_x}$. The numerator α_g measures group g's net exports (+) or net imports (-) over world gross exports (+). The denominator measures the sum of the absolute values of the price elasticities of world import demand and world export supplies. If the perfect market assumption is removed, β_g^* has to be estimated from equation (22) instead of (23). This will often result in increasing $|\beta_g^*|$ as illustrated by the two following examples, where g stands for the LDCs.

For tropical export crops $\left(\beta_g^* > 0\right)$, the difference between equations (22) and (23) is mainly due to the term in $1-\gamma_{xg}$, since α_{xg} is large and $|\alpha_{mg}|$ is small. The opportunity cost of the resources released by reducing g's exports by a marginal unit is generally substantially lower than the export price ($\gamma_{xg} < 1$). The reasons may be: - imperfect resource mobility, - market wages exceeding opportunity cost of unskilled labor, - over-evaluation of currency, - export taxes.

For most industrial products ($\beta_g^* < 0$), the difference between (22) and (23) is due to the term in $1-\gamma_{mg}$ since, this time, $|\alpha_{mg}|$ is large and α_{xg} is small. In the case of those industries for which the LDC's are over-protected by tariff barriers, the opportunity cost of reducing imports by a marginal unit would exceed the import price ($\gamma_{mg} > 1$). As for tropical export crops, equation (22) would give a higher $|\beta^*_g|$ than (23).

When the trade shares (α_j or α_g) of each decision agent are infinitely small and when the price-elasticity coefficients have non-zero values, all coefficients β are equal to zero. With these assumptions characterizing the "free competition case", each decision agent maximizes his profit by equalizing his marginal cost to the prevailing world price. When the trade shares (α_j or α_g) of some of the decision agents differ significantly from zero and when the price elasticities have finite values, the coefficients β differ from zero for some decision agents. We then depart from the "free competition case" and conflicts of interests between trading partners arise.

2.1.4 Optimum Export Tax

To compute the optimum export tax for country j, we shall assume that the resources devoted to the production of the commodity concerned are optimally allocated within that country and therefore use formula (15') for β^*_j . Since the level of the optimum export tax, expressed as the percentage e_j of the export price P_j , must be such that $\beta^*_j P_j = .01\, e_j P_j$, it follows that $e_j = 100\ \beta_j^* = \dfrac{100\alpha_j}{-\eta_m + (1-\alpha_j)\eta_{xoj}}$. The level of the tax depends therefore on the share of country j in world exports (α_j), the price elasticity of world import demand ($-\eta_m$) and the price elasticity of export supplies in countries other than i (η_{xoj}).

On the basis of this equation, the value of the optimum export tax e_j has been computed in Table 1 for selected values of the three parameters η_m, η_{xoj} and α_j. Since the marginal production cost can never become negative, country j can never reach a share of world exports exceeding the absolute value of the price elasticity of the demand for its exports. In particular, if the absolute value of the price elasticity of the world import demand is lower than unity, a country can never reach a perfect monopolistic position whatever its comparative advantage is.

The northeast corner of Table 1 remains blank because the existence of a profit maximizing country with a very high share of world exports is inconsistent with very low price elasticity coefficients. Thus, if the profit maximizing country j accounts for 90 percent of world exports and if the price elasticity of supply in the other exporting countries (η_{xoj}) is equal to 1.5, the absolute value of the price elasticity of the world import demand (η_m) should exceed .75. This can be illustrated by the historical experience of Brazil. With a 60 percent share of the world coffee exports, Brazil should have raised an export tax equal to 75 percent of the world price, if the long-term price elasticities were those shown in line (5). But exporting countries with a 3 or 10 percent share should have established taxes of only 2 or 8 percent of the world price. Although Brazil was the most efficient coffee producer, its comparative advantage vis-a-vis its competitors was not all that large. Brazil had therefore no choice but to reduce the volume of its exports when prices fell and its share shrank from 60 to 40 percent.

Brazil has established a high export tax on coffee through the mechanism of multiple exchange rates. For cocoa, the major exporters, (Nigeria, Ivory Coast and Ghana in particular) have also established large export taxes. If the long-term elasticities for cocoa were those shown in line (5), the interest of Ghana (with a 30 percent share) would be to establish an export tax equal to 27 percent of the world price according to the formula. Although the actual level of the export tax may differ from the one shown in Table 1, the fact remains that, for primary commodities with price inelastic demand and supply, major exporting countries have established substantial export taxes which are consistent with the formula.

It is important to draw a distinction between short and long-term price elasticities, especially in the case of tree crops. Thus for cocoa, while the long-term price elasticities are likely to be close to those shown in line (5), the short-term elasticities are more likely to correspond to the values shown in line (2). In the short term, the tree population being given, production can only respond to changes in the application of current inputs, such as pesticides. A number of technicians, who claim that pesticide applications are highly profitable, have criticized Ghana for having cut pesticide imports when cocoa prices fell sharply around 1965. We shall see how this behavior may be rationalized within a short-term profit maximization horizon.

Table 1: OPTIMUM EXPORT TAX FOR COUNTRY j ACTING AS A PROFIT MAXIMIZER MONOPOLIST

$$100\ \beta_j^* = e_j = \frac{100\alpha_j}{-\eta_m + (1-\alpha_i)\eta_{xoj}}$$

	Price elasticity of		α_j Country j's exports as fraction of world exports.						
	World import demand η_m	Export supply in countries other than j η_{xoj}	0	.03	.10	.30	.60	.90	1.0
(1)	- .3	+ .1	0	8	16	81			
(2)	- .3	+ .2	0	6	21	68			
(3)	- .4	+ .2	0	5	17	56			
(4)	- .4	+ .6	0	3	11	37	94		
(5)	- .4	+1.0	0	2	8	27	75		
(6)	- .4	+1.5	0	2	6	21	60		
(7)	-1.0	+ .2	0	3	8	26	56	88	
(8)	-1.0	+ .5	0	2	7	22	50	86	
(9)	-1.5	+1.0	0	2	5	18	43	82	
(10)	-1.5	+1.5	0	1	4	12	29	55	67
(11)	-2.0	+1.5	0	1	3	10	23	42	50
(12)	Infinity		0	0	0	0	0	0	0

Let us assume that, when cocoa sells at 30 cents a pound, one additional dollar's worth of pesticides gives an additional cocoa production worth $4.00. By applying pesticides, the opportunity cost of producing an additional pound of cocoa is only 25 percent of the price at which this pound of cocoa sells on the world market. With the values shown in line (2), Ghana is better off by exporting more as long as the opportunity cost of production is less than 32 percent of the world price. Consequently, when cocoa sells at 30 cent a pound, Ghana is better off when applying pesticides, even with a short-term profit maximization horizon. But when cocoa sells at only 20 cents, the opportunity cost of an extra pound of cocoa saved by pesticides reaches 37.5 percent of the world price, which exceeds the thresholds of 32 percent. If the government takes a short-term profit maximization horizon, he is better off by cutting pesticide imports.

2.2 Dynamic Game

We have discussed in the previous section the level of the optimum export tax for country j, assuming no retaliation from country j's trading partners. But if a decision taken by country A can hurt country B and vice versa, country B will try to retaliate against country A, which may, in turn, retaliate. The threat of retaliation may sometimes be a sufficient form of dissuasion to avoid an escalation which would harm both parties. Countries A and B may even decide to join in an alliance. Eventually, the main trading partners may end up in a cooperative game, called a commodity arrangement. It is generally during the course of the price downswing that the factors conducive to such an arrangement reach the required critical mass.

During the price upswing, producing countries play a non-cooperative game; each one tries to increase supply as fast as it can. When the price starts to fall, the major producing country is the first to reduce the volume of its exports, since it is the only one which can cut its losses by unilateral reduction in the volume of its exports. The bargaining power of the major producers vis-a-vis other producers may be strengthened by the possession of large stocks, as was the case of Brazil for coffee. If the price fall was temporary, the major producing country is then at the top of its strength. If the price downswing continues, rather than to carry alone the burden of supporting world prices, the main producer tries to convince other producers to join in an alliance. It will first approach the large producing countries and he may use, if necessary, the threat of flooding the market with its stocks to convince them of their gain in an alliance.

If the price elasticities are low, producers as a group always gain from a cartel restricting export supplies. The practical problem is the redistribution of the cartel's gain among its members. The most acute problem is probably that of the newcomer (N) which has a low share today but can increase its production very substantially at low cost. Let us assume that during the three years preceeding the agreement, N has exported on the average a quantity of 1, but that he could export an average quantity of 3 during the three years to be covered by the agreement. Let us further assume that the price would be 100 without agreement and 150 with agreement. while N's production cost is only 50. What is the level of the quota for which N is better off by joining the agreement?

Obviously, N is not interested in the agreement if he receives a quota equal to 1. Without agreement N would gain 3(100 - 50) = 150; with agreement he would gain only 1 (150 - 50). If (N) is offered a quota of 2, he has to make a choice between two strategies. With the first, joining the agreement, he will gain 2(150 - 50) = 200. With the second, not joining, N would gain 3(100 - 50) = 150 if no agreement is reached, but 3(150 - 50) = 300 if an agreement is established without him and if he can still sell 3 while taking advantage of the high price resulting from the agreement. N has therefore to compare the gain of 200 with a probability p_1 of gaining 150 and a probability $1 - p_1$ of gaining 300. Depending on his assessment of the probability p_1, on his attitude towards risk and on his desire for cooperation, he may choose to join or not to join. But, if N was offered a quota of 3, he would not hesitate in joining the agreement.

The situation of the major producer is different. Without agreement, he would have to reduce exports if his marginal return is lower than his marginal cost. With agreement, he will have to reduce his exports further, but he will then benefit of the reductions made by a number of other producers and, consequently, will receive more. Since the strategy "joining" is usually strictly dominant for the major exporter, the newcomer N will have an excellent bargaining power and, if he is a hard bargainer, he may succeed in pushing his quota close to 3.

If there are only a few countries like N, the major producer can convince them to join by offering them a large enough quota. But even so the cartel cannot raise prices too much, since this would attract newcomers. In some manufacturing industries, the existence of large economies of scale provides the cartel with a protection against newcomers; but, this is not generally the case for primary commodities. For those commodities, producers' cartels are therefore not very stable without the cooperation of the importing countries in enforcing the agreement. Bringing the importers in, increases the stability of the agreement, but it, obviously, reduces the scope for raising prices.

3. International Lending Agency

The case for international commodity agreements on tropical export crops is often argued as a way of raising world prices (within reasonable limits) and consequently of improving income distribution among countries. Similarly, the case for concessional lending to developing countries is generally argued on the basis of international welfare considerations. We shall consider here an international agency making project loans under concessional terms to developing countries only. In the case of projects affecting world commodity prices, we shall analyze project selection criteria in relation to international welfare. In the first section, we shall consider three simple alternative lending criteria. In the second section, we shall optimize the lending activities of the agency by maximizing an international welfare function subject to minimum income constraints for the agency's member countries.

3.1 Three Simple Criteria

3.1.1 Maximizing the Return to the Borrowing Country only

For the agency the simplest is to consider the impact of the project financed in country A on country A only. However, the application of this criterion may have very unfavorable international welfare implications, as shown by the example below.

World exports originate exclusively from the LDCs and are distributed equally among the four countries A,B,C,D. One-tenth of world exports is imported by developing country E and the remaining nine-tenths by developed countries. In relation to world prices, the elasticity η_x of the export supply in countries A,B,C, and D is equal to +.8, while the elasticity η_m of the import demand in country E and in the developed countries is equal to -.4. This set of assumptions is summarized in the first two columns of Table 2, using the notations explained in section 2.1.3.

With optimal resource allocation within each exporting country, equation (15') can be used to compute the coefficients:

$$\beta_A^* = \beta_B^* = \beta_C^* = \beta_D^* = \frac{\alpha_j}{-\eta_m + (1-\alpha_j)\eta_{xoj}} = \frac{+.25}{.4 + (1-.25).8} = +.25$$

Each exporting country taken in isolation therefore gains by implementing a project with unitary cost of production lower than 75 percent of the world price.

We shall now make the following assumptions: (a) The agency finances a "good project" for which the cost of production is only 60 percent of the world price $C_p = .6P$; (b) The country cannot implement this project without the help of the agency, because the latter brings in, not only financing, but also technical expertise; (c) The exportable supply generated by the project is only equal to .1 percent of world exports ($\alpha_p = .001$), and; (d) the transportation cost differentials T_j are negligible and $\gamma_{mj}=1$.

Table 2: IMPACT OF SMALL PROJECTS (p):

$\alpha_p = .001$ Q=1000 P=100 V_p=100

Countries	Assumptions		Net gain (+) or loss (-) π_j on projects financed by the agency	
	Price elasticities η_{xj} for exporters η_{mj} for importers $\eta = -\eta_m + \eta_x$ for world	Trade Shares α_j	Project located in A	Rounds of projects located successively in A, B, C and D
Developing countries		(+.90)	(-51.7)	(-206.8)
A exporting	+.8	+.25	+15	-60
B exporting	+.8	+.25	-25	-60
C exporting	+.8	+.25	-25	-60
D exporting	+.8	+.25	-25	-60
E importing	-.4	-.10	+ 8.3	+33.2
Developed countries	-.4	-.90	+75	+300
World Total	+1.2	.00	+23.2	+ 93.2

Taking the volume of world exports without the project as Q=1000 and the world price without the project as P=100, the value $V_p=\alpha_p PQ$ of the project output is equal to 100. The project induces a decline of world prices equal to one-twelfth of one percent, as appears from the application of equation (4):

$$\frac{-\Delta P}{P} = \frac{\alpha_p}{-\eta_m+\eta_x} = \frac{.001}{+.4+.8} = \frac{.01}{12}$$

The direct gain from the project calculated from equation (7) is 40:

$$(4) \quad \pi_p = (1 - \frac{C_p}{P}) \; V_p = .4 \times 100 = 40$$

The indirect impact $\pi_j=-\beta_j V_p$ resulting from the price decline can be calculated as follows:

$$\beta^*_E = \frac{\alpha_E}{\eta} = \frac{-.1}{1.2} = -\frac{1}{12}\,, \qquad \pi_E = -\beta^*_E V_p = \frac{100}{12} = +8.3$$

$$\beta^*_{Dd} = \frac{\alpha_{Dd}}{\eta} = -\frac{.9}{1.2} = -\frac{3}{4} \qquad \pi_{Dd} = -\beta^*_{Dd} V_p = +75$$

If the project is implemented in country A, the other exporting countries B, C and D with $\beta^*=+.25$ lose:

$$\pi_B = \pi_C = \pi_D = -.25 \times 100 = -25$$

while country A gains:

$$-\beta^*_A V_p + (1 - \frac{C_p}{P})V_p = -.25 + .4 \times 100 = +15$$

If the agency takes into account only the impact of the project in country A, it will finance the project since country A gains 15. The agency can then turn to country B and, using the same criteria, finance in B a project identical to the one previously financed in A. The agency can next move to country C and finally to country D. After this first round has been completed, the world price has declined by only one-third of one percent. On the one hand, each of the countries A, B, C and D has gained 15 once and lost 25 three times; the net loss of each country is therefore equal to 60. On the other hand, country E has gained 33.2 and the developed countries have gained 300. The agency can then proceed to a second round.

In the classical international trade model, which assumes an equilibrium or a fixed gap [4] of the trade balance, there is a self-correcting mechanism. The fall in the terms of trade for tropical export crops reduces the developing countries' import demand for exports originating from developed countries, while it stimulates the developed countries' import demand for LDC's exports. These two factors tend to reverse the initial fall in the LDC's terms of trade. However, if the initial reduction of the LDC's import capacity is compensated for by higher lending from the agency, which itself borrows its funds from the developed countries, the self-correcting mechanism is replaced by a self-generating lending process inducing an ever growing debt from the LDC's to the developed countries.

3.1.2 Maximizing the Return to the LDC as a Group

The agency maximizes this time the unweighted sum of the gains (+) and losses (-) accruing to the borrowing countries on account of the agency activities. The agency does not take into account the distribution of the total gain among the borrowing countries nor the impact on the developed countries. Assuming the agency has no constraint on the volume of its total lending, it will finance a project if the cost of production per unit of the project output C_p satisfies equation (24):

$$C_p < (1 - \beta_g^*) \; P_x$$

with P_x = world price, $\gamma_{mg} = 1.0$, $P_{xg} = P_x$

equation (22) can be written $\beta_g^* = \frac{\alpha g}{\eta} \left[1 + \frac{\alpha xg}{\alpha g} (1 - \gamma_{xg}) \; \eta_{xg} \right]$

Table 3 gives numerical values of the coefficients β_g^* under alternative assumptions. Twelve different combinations of the price elasticities are shown along the lines. Two different combinations of the trade shares associated with two different values of γ_{xg} (γ_{xg} = .5 and γ_{xg} = 1.0) are shown along the columns. The first two columns correspond to the LDC trade for cocoa. The last two columns correspond to those for wheat.

Let us consider the case of cocoa and assume that the price elasticities are those shown in line (5). If the opportunity cost of the resources released from cocoa production was equal to the prevailing world price (γ_{xg} = 1.0), the agency would finance a cocoa project only if the unitary cost of production was lower than (100-68=)32 percent of the world price. If the opportunity cost of the resources released per unit of production displaced was equal to only half of the prevailing world price (γ_{xg} = +.5) (which seems more likely), the agency would never finance a cocoa project with the price elasticities shown in line (5). With the price elasticities shown in line (6) the agency would finance a cocoa project only if the cost of production per unit of the project output was less than (100-90=) 10 percent of the world price. In practice, the agency would not finance cocoa projects.

Let us now turn to the last two columns of Table 3. This time, the LDC accounts for half of the world gross import and one-tenth of world gross exports with the price elasticities shown in line (4) (η_m= -.4 and η_x = +.6)

Table 3: Numberical Values of $100\beta^*_g$

($\gamma_{mg} = 1$, $P_{xg} = P_x$)

Elasticity in Relation to World Price		Large LDC Export Surplus $\alpha_g = .95$ $\alpha_{xg} = 1.0$ $\alpha_{mg} = -.05$		LDC Import Deficit $\alpha_g = -.4$ $\alpha_{xg} = .1$ $\alpha_{mg} = -.5$	
Imports	Exports	Opportunity Cost of Resources Released			
$\eta_m = \eta_{mg}$	$\eta_x = \eta_{xg} = \eta_{xr}$	$\gamma_{xg} = .5$	$\gamma_{xg} = 1.0$	$\gamma_{xg} = .5$	$\gamma_{xg} = 1.0$
(1) -.3	+.1			-99	-100
(2) -.3	+.2			-78	- 80
(3) -.4	+.2			-65	- 67
(4) -.4	+.6		95	-37	- 40
(5) -.4	+1.0		68	-25	- 29
(6) -.4	+1.5	90	50	-17	- 21
(7) -1.0	+ .2	87	79	-32	- 33
(8) -1.0	+ .5	79	63	-25	- 27
(9) -1.0	+1.0	73	48	-18	- 20
(10) -1.5	+1.5	57	32	-11	- 13
(11) -2.0	+1.5	49	27	- 9	- 11
(12)	Infinity	0	0	0	0

Table 4: LDC's Share in World Market for Selected Agricultural Commodities 1963-1965

	LDC's Net Exports (+) or Net Imports (-) Over World Exports $\alpha_g = \alpha_{xg} + \alpha_{mg}$	LDC's Exports Over World Exports α_{xg}	LDC's Imports (-) Over World Exports α_{mg}
Cocoa	.95	.99	-.04
Coffee	.93	.98	-.05
Sisal	.92	.95	-.03
Abaca	.91	.97	-.06
Bananas	.90	.94	-.04
Copra	.84	1.00	-.16
Groundnuts	.82	.90	-.08
Natural rubber	.79	.97	-.18
Palmoil	.79	.96	-.17
Jute	.77	.95	-.18
Tea	.74	.96	-.22
Coconut oil	.74	.89	-.15
Linseed oil	.74	.84	-.10
Groundnut oil	.63	.82	-.19
Fishmeal	.57	.62	-.05
Sugar	.55	.785	-.23
Cotton	.44	.64	-.205
Maize	.15	.245	-.09
Timber	.10	.18	-.075
Linseed	.10	.12	-.015
Lamb	.02	.06	-.04
Rice	-.08	.70	-.78
Wheat	-.44	.10	-.54

and $\gamma_{xg} = +.5$, the agency would finance a commodity project as long as the unitary production cost does not exceed the world price by more than 37 percent. With $\gamma_{xg} = +1.0$, the threshold price would increase only from 37 to 40 percent.

The crucial importance of the LDC trade share appears most clearly if we assume $\gamma_{xg} = \gamma_{mg} = 1.0$, since in this case

$$\beta_g^* = \frac{\alpha_g}{\eta}$$

By maximizing the return to the LDC as group, the agency would draw a black list of commodities. Table 4 shows that for 16 of the 23 agricultural commodities listed, the coefficient α_g is higher than half. Most[1/] of these (which account for the bulk of LDC's agricultural export earnings) would be on the black list, few agricultural commodities for exports but most of the industrial products would be on the white list. The lending implications would be clear-cut because the grey area does not contain many products having a large weight in LDC trade.

3.1.3 Maximizing the Weighted Sum of National Gains

Taking the same weight for all developing countries and a zero weight for all developed countries is equivalent to the second criterion. It leads to ranking commodities k according to a scale vector β_k. Using weights negatively correlated with average income per capita leads to establishing a matrix β_{kj} by commodity and by country. The agency could then finance a cocoa project in country A which is very poor, but not in country B which is not so poor. If country B had nothing but a cocoa project to submit to the agency, country B would have made net loss on account of the agency lending activities.

1/ The coefficient β_g depends not only on the share α_g but also on the price elasticity coefficients η and the opportunity cost parameter γ_g. If natural and synthetic rubber were perfect substitutes, the coefficient β_g should be computed by treating natural and synthetic rubber as a single product.

This last criterion might provide an acceptable rule of thumb to deal with most commodity producing projects. It is however worth analyzing the problem of interactions among countries within an optimizing model, since it raises more general problems associated with the allocation of scarce international funds.

3.2 Programming Model

The objective function reflects the mission assigned to the agency by its members collectively. Since the agency would not be able to perform its mission satisfactorily without the individual cooperation of its members, lower bounds on the gains of each member on account of the agency's activities are introduced in the model. The first section outlines the structure of the model. The second considers the implications regarding lending criteria for projects located in country A and having an impact on countries other than A.

3.2.1 Structure of the Model

Table 5 outlines a static model optimizing the agency's lending activities over a single period. A distinction is made between the developed member-countries i to which the agency cannot lend and the developing member-countries j to which the agency can lend; no account is taken of the non-member countries. Developed countries i can be affected by projects financed by the agency only indirectly through price effects. Developing countries j can be affected both directly and indirectly. The impact of the agency's activities on a given country (i or j) is measured by the present value (Y_i or Y_j) of the discounted stream of gains and losses incurred by that country on account of all the loans extended by the agency to all countries j.

The objective function shown in row (11) is the weighted sum of the gains and losses Y incurred by each member country. The weights (w_i and w_j) are a decreasing function of the national per capita income which each country

Table 5: Lending Model

	Columns	Subsidy	Lending Activities	Production of Commodity k	Gain (+) or Less (-) of		R.H.S.
					Developing Country j	Developed Country i	
	Rows	S_{jt}	X_{jts}	V_k	Y_j	Y_i	
(1) Step constraints reflecting decreasing returns to capital	X_{jts}		$+X_{jts}$				$\leqslant \bar{X}_{jts}$
(2) Production of commodities k affecting world prices	V_k		$+\sum_j v_{kj} \sum_s \delta_{kjs} X_{kjs}$	$-V_k$			$=0$
(3) Gain of developing country j on account of the agency's activities	Y_j	$\sum_t S_{jt}$	$+\sum_t \sum_s \delta_{jts} (r_{jts} - r_h) X_{jts}$	$-\sum_k \beta^*_{jk} V_k$	$-Y_j$		$=0$
(4) j's minimum gain					$+Y_j$		$\geqslant \bar{Y}_j$
(5) Minimum gain for project t in j	Y_{jt}	S_{jt}	$+\sum_s \delta_{jts} (r_{jts} - r_h - \beta^*_{jt} v_{jt}) X_{jts}$				$\geqslant \bar{Y}_{jt}$
(6) Gain or loss to developed country i on account of the agency activity	Y_i			$-\sum_k \beta^*_{ik} V_k$		$-Y_i$	$=0$
(7) i's maximum loss						$+Y_i$	$\geqslant \bar{Y}_i$
(8) Subsidy to j	S_j	$\sum_t S_{jt}$	$-(g_\sigma - g_h) \sum_t \sum_s X_{jts}$				$\leqslant 0$
(9) Maximum subsidies	S	$\sum_j \sum_t S_{jt}$					$\leqslant \bar{S}$
(10) Maximum lending	X		$+\sum_j \sum_t \sum_s X_{jts}$				$\leqslant \bar{X}$
(11) Objective function	Y				$+\sum_j w_j Y_j$	$+\sum_i w_i Y_i$	MAX

r_{jts} rate of return on investment

r_h rate of interest on hard loans

g_σ grant content of soft loans

g_h grant content of hard loans

v value of exportable supply per dollar lent

δ conversion from streams into present values

w welfare weights

β* coefficient defined in section 2.1.3

k subscript for commodities affecting world prices would have reached in the absence of the agency. The minimum gain ($\overline{Y}_j > 0$) to be insured to a given developing country j appears as a constraint in rows (4). The maximum permissible loss ($\overline{Y}_i < 0$) to a given developed country i appears as a constraint in rows (7).

The agency can lend in the form of hard (h) and soft (σ) concessional loans. The grant contents g_h and g_σ of these two types of loans are defined in relation to the terms of non-concessional loans available to the developing countries in unbounded quantities. The coefficient g_h measures the difference between the present value of the stream of repayments per dollar of non-concessional loan and the present value of the stream of repayments per dollar of concessional loan made by the agency on hard terms. By definition, the grant content is greater for soft than for hard concessional loans ($g_\sigma - g_h > 0$).

The agency extends to country j the mix ($X_{\sigma j}$, X_{hj}) between soft and hard loans. This mix is translated in the model's language by the mix between the total volume of lending X_j ($= X_{\sigma j} + X_{hj}$) on hard terms and the volume of straight subsidies S_j ($=(g_\sigma - g_h)\ X_{\sigma j}$). The agency can modify the allocation of subsidies among countries only by modifying the country mix between hard and soft loans ($X_{hj}/X_{\sigma j}$). The level of the subsidy activity S_j allocated to country j is therefore constrained by $0 \leqslant S_j \leqslant (g_\sigma - g_h) X_j$. The upper bound of this constraint is introduced in rows (8). The overall financial constraints imposed on the agency in terms of total lending $\overline{X}$ ($= \overline{X}_h + \overline{X}_\sigma$) and total subsidies $\overline{S}$ $\left[= (g_\sigma - g_h)\ \overline{X}_\sigma\right]$ are shown in rows (10) and (9).

The agency is assumed to be the only one extending concessional loans to developing countries. Its lending activities are fully defined by the matrices S_{jt} and X_{jt} characterizing respectively the allocation of subsidies and the volume of lending by countries j and by types of project t. →

Among projects t, a distinction is drawn between projects k which affect world commodity prices (for example, a cocoa project) and projects f which do not (for example, an education project).

Let us consider the projects f first and call r (X_{jf}) the rate of return of the foreign capital inflow X_{jf} in relation to the value of the objective function of country j, defined as the combined consumer-producer surplus of that country. If we approximate the curve r (X_{jf}) by the step function shown in figure 7, the present value of the increase in the country's objective function resulting from the investment X_{jf} can be written:

$$(7') \quad \Pi_f^{(j)} = \sum_s \delta_{jfs} (r_{jfs} - r_h) X_{jfs}$$

The subscript s characterizes the step of the staircase shown in figure 7. The parameter r_h measures[1] the interest rate on hard agency loans and the coefficient δ_{jfs} converts the stream of benefits and costs into its present value.

---

[1] Let us call ρ the rate at which the country discounts gains or losses and let us characterize the repayment flow for a hard agency's loan by the value x_τ to be repaid in year τ. This hard agency loan has the same present value that a loan of X with a constant yearly charge of $r_h X$ for interest and indefinite repayment period for the principal. The rate r_h is given by:

$$r_h = \rho \sum_{\tau=1}^{\tau=\infty} \frac{x_\tau}{(1+\rho)^\tau}$$

Since $\sum_{\tau=1}^{\infty} \frac{1}{(1+\rho)^\tau} = \frac{1}{\rho}$, it can be easily verified that the equality applies if $x_\tau \equiv r_h$. For an investment X generating a steady stream of value added V, the rate of return on the investment is $r = \frac{V}{X}$ and the coefficient δ is $\frac{1}{\rho}$. If $r = r_h$, the profit Π is equal to zero.

If $r > r_h$, the profit Π is $\frac{r-r_h}{\rho} X$.

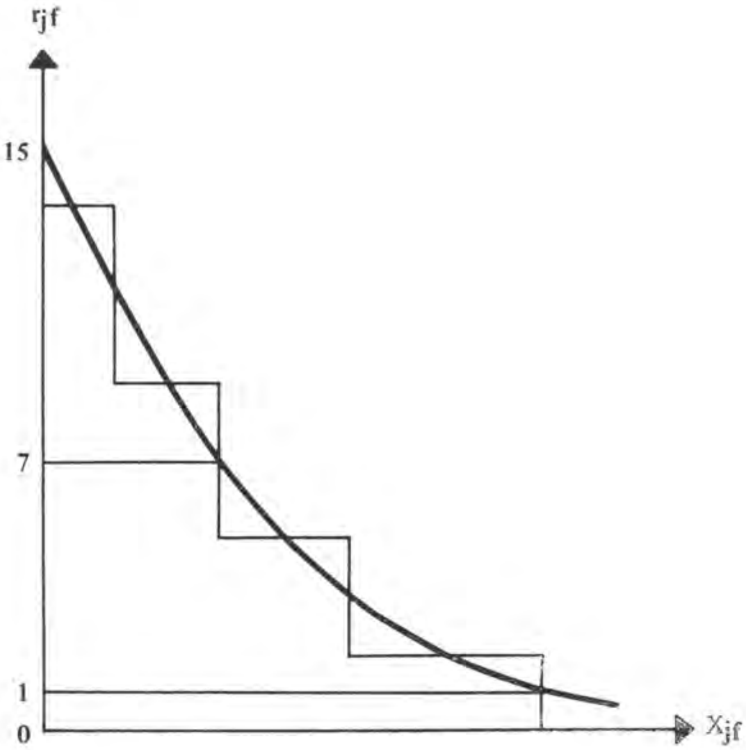


Figure 7: DECREASING RETURN TO FOREIGN CAPITAL INFLOW

Let us now turn to projects k. The impact on country j of all projects k financed by the agency results from the additions of the two components defined by equations (7) and (11) in pages 29 and 30. The first $\Pi_k^{(j)}$ is the direct gain accruing to country j on account of the projects k implemented in country j. The second Π_{jk} is the indirect gain (or loss) accruing to country j on account of the price effect induced by the project k financed by the agency in all countries including j.

The first component $\Pi_k^{(j)}$ is computed for projects k in the same manner as for projects f. The direct gain from all project f and k financed by the agency in country j is obtained, therefore, by summing equation (7') over all projects t (f and k), as shown in line (3) column (X) of table 5. The step[1] constraints $X_{jts} < \overline{X}_{jts}$ characterizing the decreasing return to foreign capital inflow in country j appear in rows (1).

[1] The number of rows (1) could be reduced to one per type of project in country j by following a device similar to the one described for the demand curves in footnote 1, p. 15.

The second component π_{jk} is equal to $-\beta^*_{jk} V_k$ where β^*_{jk} is the approximation of β for small projects given in equation (13) and V_k is the discounted export value generated by all projects k financed by the agency in all countries (j included). The values of V_k are computed in rows (2) by multiplying the volume of the agency's lending X_{jk} by the coefficient v_{kj} characterizing the value of exportable supplies generated in country j per dollar lent and by the coefficient δ_{jks} transforming streams in present values. The program remains linear by measuring v_{kj} in relation to the price which would have prevailed without the agency's intervention and by using the approximated formula for β^*. These approximations are justified[1/] for two reasons. First, for commodities on the black list (coffee, cocoa, tea, etc.), the agency will finance very few projects by applying the model outlined here. Second, for the commodities on the white list (wheat, industrial products, etc.), the agency is unlikely to have a large impact on world prices, because developing countries account for a small part of world output.

The net gain of countries j is computed in rows (3) by adding up the subsidy S_j, the direct gains $\pi_k^{(j)}$ for all projects financed in j and the indirect gains (or losses) π_{jk} incurred by j on account of all the projects k financed by the agency in all countries. Rows (4) insure that countries j will gain at least $\overline{Y}_j$ on account of all the lending operations of the agency. Rows (5) insure that each project t implemented in j will bring to j a gain at least equal to $\overline{Y}_{jt}$. If all the $\overline{Y}_{jt}$ were equal to zero, rows (5) would express that no project can be unprofitable for the country which has to implement it.

For developed countries i, rows (6) adds up only the indirect gains (or losses) on account of the projects k financed in all countries j. Rows (7) insure that i's net loss cannot exceed $-\overline{Y}_i$.

[1/] If it were not β would be approximated by a staircase.

3.2.2 Implications

To comprehend the lending implications of the model, it is convenient to start from the simplest case discussed below under A1 and to go progressively to the most complex case B2, the structure of which was outlined in the previous section.

A. No Externalities ($\beta_g \equiv \beta_j$)

In the absence of commodities k, rows (2), (5), (6) and (7) as well as columns V_k and Y_i disappear from Table 5. The distinction among types of projects t becomes irrelevant. The agency needs only to know the curve of Figure 7, which defines for all types of investments the marginal efficiency of foreign capital r (X_j) in each potential borrowing country j.

A.1 Efficiency Only ($w_j \equiv 1$, $\delta_j = \delta$, no $\overline{Y}_j$ constraint)

Since the distribution of the subsidy S among countries j does not affect the value of the objective function $Y = \sum_j Y_j$, the agency can use for every country the same blend between soft and hard loans and therefore the same lending rate $\overline{r}_d$. If r_j is the marginal return to country j, the contribution of a variation d X_j in the amount lent to j is d $Y = \delta\ (r_j - \overline{r}_d)$ d X_j. In the optimal solution, the ratio dY/dX_j, and consequently the marginal rates of return r_j, must be identical for every country j.

The common cut-off rate of return r is the ordinate of the point where the curve r (X) (aggregated over all projects t and all countries j) intersects the vertical of abscissa $\overline{X}$ (maximum amount which the agency can borrow). If the agency could borrow unlimited amounts at a fixed rate of interest rate $\overline{r}$, the model would become redundant. Financing any project with a rate of return $r_j \geqslant \overline{r}$ would always be optimal. 1/

1/ This may not be true in dynamic models due to future repayment constraints, which reflect the limited ability of the country to earn (or save) more foreign exchange in latter years.

A.2 Efficiency versus Equity ($w_j \neq 1$, no $\overline{Y}_j$ constraint)

With a single lending rate $\overline{r}_d$, the marginal utility of lending to country j is $dY/dX_j = \delta_j\, w_j\, (r_j - \overline{r}_d)$. In the optimal solution, the cut-off rate of national return for country j is $r_j = \overline{r}_d + \frac{1}{\delta_j w_j} \frac{dY}{dX}$. The cut-off rates should therefore be higher in the rich than in the poor countries.

For a given grant content, the agency can raise Y by differentiating its terms of lending. Let us consider a poor country A and a rich country B with $w_A - w_B > 0$ and $\delta_A w_A - \delta_B w_B > 0$. Subsidies must be allocated by priority to A, since, in Y, S_A is more heavily weighted than S_B. Assuming all subsidies go to A, the optimal lending allocation between A and B requires $r_B - r_A = (\frac{1}{\delta_B w_B} - \frac{1}{\delta_A w_A}) \frac{dY}{dX}$. As was the case before with the single lending rate $\overline{r}_d$, with differential lending rates the cut-off rates should still be higher in the rich than in the poor countries.

A.3 Efficiency $(r_j - \overline{r}_h)$ versus Equity (w_j) with National Income Constraints $\overline{Y}_j$

The shadow prices of the binding income constraints $\overline{Y}_j$ measure the international welfare cost of satisfying individual countries. The prices obtained in the first iteration would help in adjusting the $\overline{Y}_j$ in relation to the institutional weights of those countries.

B. Externalities ($\beta^*_{gk} \neq \beta^*_{jk}$)

Let us turn to the case of projects producing commodities k for which the world import demand and the world export supply are price inelastic. The indirect price effects of project k financed by the agency in country j on countries other than j enters now in Y.

B.1 Developing Countries as a Group ($w_j \equiv 1$, $w_i \equiv 0$, $\delta_j \equiv \delta$)

The difference between the contributions of the loan dX_{jk}

to the value of the objective function Y_j of country j where the project is located and to the value of the international objective function Y is given by:

$$d\,Y_j - d\,Y = \delta\, v_{jk}\, (\beta^*_{gk} - \beta^*_{jk})\, (d\,X_{jk}) = \delta(r_{jk} - r_k)\, (d\,X_{jk})$$

where β^* is the coefficient defined in section 2.1.3, δ a coefficient transforming streams into present values and v_{jk} the output/capital ratio (annual value of the exportable supply generated by the project divided by the value of the capital lent by the agency).

The difference between r_{jk} (rate of national return to country j) and r_k (rate of international return to the agency) is:

$$r_{jk} - r_k = v_{jk}\, (\beta^*_{gk} - \beta^*_{jk})$$

This difference can be illustrated numerically by considering a cocoa project implemented in a marginal exporting country j, for which therefore $\beta^*_{jk} = 0$. With a capital/output ratio equal to 2, $v_{jk} = .5$; for $\beta^*_{gk} = 1$ (see p. 49), $r_{jk} - r_k = .5$. The rate of international return of this cocoa project would then be 50% lower than the rate of national return to the country where the project is located.

B.1.1 <u>The Commodity Black List</u> ($\overline{S} = 0$, no $\overline{Y}_j$ income constraints)

Let us assume a 10% cut-off rate for international returns. Among projects f which do not induce externalities, the model would select those which have a rate of national return r_{jf} larger than 10%. But, the cocoa project described above would be selected only if its rate of national return r_{jk} were higher than 60%.

It would not be very sensible for the agency to finance projects with such high rates of national return. If such projects were not financed by the agency, they would be implemented by country j from other financial sources. Due to the fun i ility of capital, these agency loans would ultimately be used to implement other projects, which would remain unknown to the agency.

Without using a model, the agency could draw a commodity black list including all commodities k for which the difference between the national and international rates of return exceeds the acceptable threshold (for example, 15%).

B.1.2 Selected Trade Strategy ($\overline{S} > 0$, $\overline{Y}_j$ and $\overline{Y}_i$ income constraints)

In the previous case, the agency avoided to deteriorate the LDC's terms of trade by not investing in particular fields. Now, the agency can contribute to improve the LDC's terms of trade by subsidizing investments for which the international rate of return exceeds the national rate of return.

Let us consider the case where the agency has a choice in country j between a cocoa project C (with $\beta^*_{gc} > 0$) and a pulp and paper project P for which the LDC's have a large and growing import deficit (with $\beta^*_{gp} < 0$). Due to price externalities, the rate of international return is higher for P than for C ($r_P > r_C$). If the rate of return to country j is higher for C than for P ($r_{jP} < r_{jC}$), there is a conflict. Let us consider the following case:

$r_{jP} < r_{jC} < \overline{r}_j$ $\quad$ $\overline{r}_j$ = interest rate at which country j can borrow commercially

$\overline{r}_\sigma < r_{jP} < \overline{r}_h$ $\quad$ $\overline{r}_\sigma$ and $\overline{r}_h$ = interest rates on soft and hard agency loans

$r_C < r < r_P$ $\quad$ r = cut-off rate of international return

The agency prefers P to C. But country j would select C, if it were to receive the same terms of lending for P and C. Lending for P on soft terms is a way of solving the conflict. Since $r_{jP} - \overline{r}_\sigma > 0$, country j is better off with than without project P. The model optimally allocates the subsidies S_{jk} by projects and countries by maximizing Y, while giving to each country, through the constraints $Y_{jk} \geqslant \overline{Y}_{jk} > 0$, enough incentive for implementing projects k.

Let us now assume that the agency has no choice among projects. For example, the agency cannot find in Burundi anything but a coffee project. To fulfill Burundi's minimum income constraint, the agency has to finance the coffee project. Minimizing the size of the project by financing it with soft loans will then generally be optimal.

B. 2. Efficiency (r_{jt}) Versus Equity (w_j), Policy Considerations ($\bar{Y}_j$) and Externalities ($\beta^*_{gk} - \beta^*_{jk}$).

This case is the most general one outlined in Table 5. To maximize the value of its objective function , the agency has to allocate subsidies according to a matrix S_{jt} and to select projects according to the matrix r_{jt} defining the cut-off rates of national returns by countries j and types of projects t.

This two-way classification by income levels and by types of projects is consistent with the allocation of public funds within a Welfare State. On the one hand, subsidies are given to low income groups while high income groups are taxed. On the other, subsidies are given to sectors such as education and health where the social return exceeds the private return, while taxation is imposed on polluting industries where the social cost exceeds the private cost.

Short of taking into account the interaction between these two types of criteria and short of building up the full matrix r_{jt}, the Welfare State can use the two criteria independently. It can extend soft loans to individuals or regions falling below a given income level and to sectors for which the social return substantially exceeds the private return. Similar simplifications could be made by the international lending agency. Thus, short of computing the matrixes S_{jt} and r_{jt}, the agency could draw a black list of the types of projects not to be financed even with hard loans and a white list of the types of projects to be financed with soft loans. Such a list would not be used as the bible but as "a presumption."

A differentiation by types of projects is not trivial for an agency lending to all developing countries and only to developing countries. Most countries at an early stage of development have fairly similar trade patterns and, for many of the export products common to those countries, the price-elasticities of the import demand of the developed countries is low. This clearly applies to tropical non-competing commodities, such as coffee, cocoa and tea. It may also apply to a number of competing manufactured products with a high labor content, such as cotton textiles and clothing, because developed countries may unfortuntely impose quota restrictions to protect their "depressed domestic sectors", once the level of imports exceeds a critical mass. By measuring the coefficients β in relation to a forward projection of international trade, the agency could integrate an LDC's trade strategy in the criteria for project selection.

The case of commodity projects was used in this paper to illustrate the impact of a given type of external economies. Research applied to the particular conditions prevailing in a number of developing countries could provide another illustration.

The research on improved wheat and rice varieties, conducted respectively in Mexico and the Philippines is a classical example of external economies. Research on capital-labor substitutions could be another promising area. For a single labor-surplus country, it may be more profitable to borrow industrial technologies from the shelves of the rich countries than to adjust it to local conditions. But for the international community it may be worth using international subsidies to develop technologies better fitted to the resources endowments prevailing in the LDCs.

BIBLIOGRAPHY

[1] Ian M. D. Little and James A. Mirlees, Manual of Industrial Project Analysis in Developing Countries, Vol. II. OECD Development Center, Paris, 1969.

[2] J. R. Hicks, "Value and Capital", London, 1965, Note to Chapter II Consumer's Surplus, pp. 38-41.

[3] E. J. Mishan, "What is Producer's Surplus?" American Economic Review. December 1968, pp. 1269-1282.

[4] M. C. Kemp, "The Gain from International Trade", The Economic Journal, Dec. 1962, pp. 803-819.

[5] P. A. Samuelson, "The Gains from International Trade Once Again", The Economic Journal, December 1962, pp. 820-829.

[6] R. Duncan Luce and Howard Raiffa, Games and Decisions: Introduction and Critical Survey, New York, J. Wiley, 1957.

[7] H. B. Chenery, "The Interdependence of Investment Decisions" in "Readings in Welfare Economics", 1961, pp. 336-371.

[8] M. Bruno, "The Optimal Selection of Export-Promoting and Import Substituting Projects", Planning the External Sector: Techniques, Problems and Policies, New York, U.N., 1967.

[9] L. M. Goreux, "Price Stabilization Policies in World Markets for Primary Commodities: An Application to Cocoa", IBRD, 1970.

[10] A. S. Manne, "Lending Criteria for an Ideal International Development Agency", IBRD, Jan. 1971.

[11] I. M. D. Little

[12] J. H. Duloy and R. D. Norton "Agricultural Sector Model" in "Project Decisions and Multi-Level Planning: Case Studies in Mexico", edited by L. M. Goreux and A.S. Manne, North Holland Publishing Company, to be published in 1972.

[13] A. Condos, L. M. Goreux, R. Vaurs, "An Agricultural Model from the Ivory Coast Programming Study", October 1971, Development Research Center, IBRD.

December, 1971 DRC file Draft

CHAC, A Programming Model of Mexican Agriculture*

John H. Duloy and Roger D. Norton

Contents Page,

* This is to form a chapter in the forthcoming volume tentatively titled "Project Decisions and Multi-Level Planning: Case Studies in Mexico".

DRAFT
November, 1971

CHAC, A Programming Model of Mexican Agriculture*

John H. Duloy and Roger D. Norton

1. Introduction

Agriculture in Mexico, as in most developing economies, is a major source of employment and foreign exchange earnings. About half of the Mexican labor force is agricultural, and, directly and indirectly through both raw and processed products, agriculture accounts for almost half of export earnings. The crucial role played by agriculture in the economic development of Mexico is widely acknowledged**. It was partly

---

*

This chapter may be viewed as a progress report on a long-term study of Mexican agriculture sponsored jointly by the Mexican Government and the World Bank. The model described here is now operating in the Ministry of the Presidency in Mexico and it will go through several more phases of refinement there; the authors are grateful to Lic. Leopoldo Solís for his encouragement and continuous support and his patience during the long gestation period of this model. The agricultural study was conceived by Louis Goreux and Alan Manne, who have given us useful comments and criticism throughout, and it has drawn upon the earlier work of Dr. Luciano Barraza, now of the Mexican Ministry of Agriculture and Livestock, with programming models of individual irrigated districts in Mexico. Dr. Barraza also has provided helpful comments on several aspects of the study. The enormous burden of constructing most of the 80,000 coefficients in the model has been borne with unflagging energy and goodwill by Lic. Luz María Bassoco, of the Presidency, and Lic. Teresa Rendón, of the Bank of Mexico. Mr. Gary Kutcher of the World Bank became a (nearly prostrate) human link between the model and the computer, and gave useful comments on the model's design.

**

See, for example, L. Solís (1971).

because of this pivotal role that agriculture was chosen as one element in the multilevel planning study: It offers an opportunity to explore relationships between sector policies and economy-wide development strategies. Agriculture also provides an example of strong linkages between investment project decisions and sector-level policies because agricultural trade policies and policies on pricing inputs and products significantly affect estimated project rates of return.

Therefore, the initial aim in constructing the agricultural model, CHAC*, was to formalize the major aspects of micro-level and sectoral decision-making. In keeping with the orientation of this volume, the broad theme of the agricultural study is linkages between different levels of decision-making, but, as is usual in the case of large-scale model building exercises, there is more than one underlying purpose.

Aside from the multi-level aspects, the sector study has been designed to serve both the Mexican Government's interest in analytic tools for planning sectoral policies, and the World Bank's interest in the methodology of project appraisal techniques and in general policy planning models. As a tool for policy makers, CHAC is designed to be addressed to questions of pricing policies, for both inputs and outputs, trade policies, employment programs, and some categories of investment allocation. It is not particularly well suited for analyzing agricultural research and extension programs, crop insurance policies, or credit policies. It is structured so that it is a simple matter to change factor prices, including costs of labor, capital, water, and agricultural chemicals, and to represent subsidies to production by crop and geographical area. The prices received by farmers and paid by consumers for internationally tradeable commodities also may be adjusted readily to reflect tariff, taxation, and exchange rate policies.

---

* CHAC takes its name from a rain god of the Maya.

Commodity demand functions are included within the structure of CHAC, and hence prices are truly endogenous.* Since relative product and factor prices are the dominant policy instruments in agriculture, this feature of the model gives scope for a wide variety of policy experiments.

Employment in agriculture is one of the major focuses of CHAC. The model is designed to yield the seasonal employment patterns for farmers and hired labor, in twenty different geographical areas. Accompanying parametric solutions are conducted to estimate the scope for labor-capital substitution. The model's structure makes it possible to divide labor absorption possibilities into three categories: those arising purely from the technology set, those arising from changes in the product mix in domestic consumption, with consequent changes in relative product prices, and those arising from changes in the international trade mix.

The production side of the model is decomposable into submodels for each of the twenty geographical areas. Under appropriate assumptions on prices, each submodel has been solved as a separate model. One of them has been formulated with a wider array of farm types so that it may serve as the "project model". It was first solved in isolation to estimate a rate of return schedule for potential fixed investments, and then it was incorporated into the sector model to see how interregional comparative advantage affects the estimated benefits to investment in one area.

The initial version of CHAC reported here covers only short-cycle crops.** Tree crops, livestock, forestry, and fisheries have been excluded. There is significant interdependence between the short-cycle crops and livestock, through forage production and pricing and through allocation of labor

---

* Some programming models yield marginal costs of production, or "supply prices", but in the absence of demand functions these do not yield equilibrium prices. See, e.g., Heady, Randhawa, and Skold (1966), and Piñeiro and McCalla (1971).

** Annuals plus sugar cane and alfalfa

and capital, and there is competition with some long-cycle crops for land and other resources. Nevertheless, it was decided to limit the scope of the first version in the interests of a more thorough treatment. Work is underway on companion models of tree crops and livestock which may be solved independently and also may be linked with CHAC for some purposes.

CHAC is a large-scale model, some 1,500 equations* in the initial version, so it is important to assess the possibilities for aggregation. Some experiments on aggregation have been carried out, and more are planned. Aggregation effects are discussed for the case of the project model in the following chapter, and the linkage with DINAMICO is carried out on the basis of an aggregated version of CHAC. However, as of this report aggregation effects had not been very well explored; further research is scheduled in this area.

2. Overview of the Model: Types of Results

CHAC is a sector-wide model in the sense that it describes total national supply and use - production and imports, domestic demand and exports - for the thirty-three principal short-cycle crops in Mexico.** It is a one-period model, so the timing of investment decisions cannot be studied, but investment choices are included in the model. On the demand side, consumer behavior is assumed to be price elastic, and thus market-clearing commodity prices are endogenous to the model. As noted, on the production side it is decomposable into twenty submodels representing different geographical entities.

The crops included in CHAC represent about 48% of the value of production in the agricultural sector of DINAMICO. The other components are fisheries, forestry, livestock, and long-cycle crops. The implications of this difference in coverage for the CHAC-DINAMICO link is discussed in chapter 14.

---

* and 80,000 nonzero coefficients.

** These thirty-two crops represent more than 99% of the value of production of short-cycle crops.

Basically, CHAC has been structured from a viewpoint of micro-economic theory rather than macro-theory. One reflection of this is the level of disaggregation: individual commodities and seasonal inputs are treated. A less trivial reflection of the micro-economic orientation is the fact that the model describes a particular form of market equilibrium, in terms of prices and quantities, with corresponding representations of producer and consumer behavior. In the solutions reported in this study, the market form described is the competitive one, but the same structure can be utilized to represent the sector as a monopolistic supplier of agricultural products. Purely as a descriptive matter, the competitive market mechanism is closer to the actual processes which determine production and prices in Mexican agriculture, and, therefore, it is adopted as the basis for the model. Government policies, such as price supports, import quotas, and input subsidies and their impacts on producers' incomes, employment, and other variables, are evaluated as interventions in a basically competitive market.

The same spirit is found in the treatment of factor markets, although they are specified in less detail. Simple labor supply functions are specified, based on observed wages by region, and the labor market equilibrium is competitive, and not monopsonistic or monopolistic. The services of hired labor are assumed to be offered at observed market wages, in the absence of policy instruments which explicitly affect wages. Experiments have been made with different assumptions on the short-run reservation price of farmers, which is no doubt less than the market wage because they are acting with a view to the income stream over a longer term. But it has not been assumed that farmers - or any others - offer their services at a zero wage. Similarly, purchased inputs and services of machinery and draft animals are priced at observed market prices, except for those experiments in which input prices are explicitly subsidized or taxed.

In all solutions the same maximand is used, the sum of producers' and consumers' surpluses. This insures the competitive market equilibria in the optimal solution. The model is not solved under, say, an employment maximand, for it is not clear which policy instruments, if any, could implement such a solution. Rather, the employment implications of specific policy changes are simulated. When policy changes are involved, the instruments are made explicit in the model, and thus all of the sectoral implications of a policy change can be estimated.

Jmy?

A brief mention of a couple of the qualitative results of the model will help illuminate the essential points of its structure.* When solved for different points in time, the model shows a significant social return to investment but virtually no private benefits from the investment. Farmers' per capita incomes tend to remain stagnant while production is increasing at roughly the historical rate. This phenomenon is not inconsistent with the experience of many countries: the process of development has been accompanied by ever-increasing budgetary outlays on agricultural subsidies in order to counter adverse market pressures on incomes. In the model, this is a reflection of the assumption that demand for most agricultural products is relatively price-inelastic, combined with competitive market structure.** It is indeed optimal for atomistic, price-taking producer to invest, although in the aggregate this may make producers worse off.

---

* A systematic review of the results is given in chapter 13.

** Producers' incomes are measured endogenously in the model by means of the monopolistic demand structure; see pp below

Another result, derived mainly from the project model but supported by the results of the sector model, is that labor-capital substitution possibilities are significantly stronger when changes are permitted in the domestic consumption and trade mixes. In other words, an effective employment-absorption program in agriculture must be accompanied by trade policy changes and/or changes in relative prices in the market for agricultural products. In order to address this kind of question properly, a wide range of technological choices in production were included in the model, along with domestic demand and trade activities. There are more than 2,300 different production techniques for 32 crops in 20 districts,* ranging from completely non-mechanized to completely mechanized, and including different degrees of efficacy in irrigation as well as non-irrigated techniques.

These points and other aspects of the structure are discussed more fully in the following sections.

3. Basic Structure of the Model

Separation of sources of supply and demand, for both products and inputs, is the basic rule under which CHAC is specified. For each crop, there are production activities differentiated by location and technique, and, for 21 of the 32 crops, there are importing activities. There are corresponding activities for sales on the domestic and export markets. Effectively the model contains multiple-step supply and demand functions for each crop, and these functions for different crops are interdependent. For most crops, the sector-wide supply function contains

---

* Chapter 11 describes the formulation of these technological alternatives.

dozens of steps, and in some cases there are more than one hundred steps. The demand formulation is flexible: it permits arbitrarily close approximation to a nonlinear form, in the event of full information on the demand functions, but it also may be based on very limited information on demand. Commodity balance equations require the clearing of markets, with simultaneous determination of equilibrium prices and quantities. Commodity prices are either completely endogenous (for non-traded goods) or endogenous between import and export prices. The assumed import and export prices may be varied in alternative solutions to reflect different world market conditions and tariff policies, and sets of prices may be fixed in order to investigate the effects of price support policies.

The incorporation of demand functions, instead of exogenous product prices, provides a more realistic description of aggregate market conditions faced by farmers, and it reduces the tendency of such programming models to seek solutions with extreme crop specialization. It also opens the door to investigation of alternative forms of market equilibrium. Under appropriate objective functions, the same model may simulate a sector which behaves either as a monopolistic supplier of products or as a collection of competitive producers. And by casting one of the two objective functions in the role of a constraint, it is possible to explore the trade-offs and complementarities between producers' and consumers' welfare.*

---

* These properties of the model are developed more fully below.

Table 10-1 Area Harvested and Value of Production of the Crops in CHAC

(1966/1967)

Crop*	Irrigated Area Harvested (ha.)	Irrigated Value (000 Pesos)	Non-irrigated Area Harvested (ha.)	Non-irrigated Value (000 Pesos)	Total Area Harvested (ha.)	Total Value (000 Pesos)
Maize	441,939	826,407	7,844,996	7,681,953	8,286,935	8,508,360
Cottonfibre	415,997	2,521,700	279,382	887,876	695,379	3,409,576
Sugar Cane	85,280	445,141	402,318	1,572,298	487,598	2,017,439
Beans	45,569	109,628	2,194,453	1,704,005	2,240,022	1,813,633
Wheat	421,685	1,113,180	304,910	304,050	760,595	1,417,230
Sorghum	268,037	487,479	307,823	414,040	575,860	901,519
Green Alfalfa	24,784	151,465	83,347	602,982	108,131	754,477
Tomatoes	18,713	641,201	26,533		45,246	590,273
Rice (unhulled)	58,482	211,843	94,160	206,916	152,642	418,759
Sesame	36,349	71,281	215,760	278,877	252,109	350,158
Safflower	100,679	201,768	64,254	126,633	164,933	328,401
Strawberries	3,371	45,412	5,454	282,849	8,825	328,261
Tobacco	2,348	25,913	37,260	277,127	39,608	303,040
Watermelon	3,272	20,669	30,228	258,637	33,500	279,306
Potatoes	3,428	33,550	30,854	228,382	34,282	261,932
Green Chile	5,975	57,337	36,527	150,417	42,502	207,754
Chickpeas	28,533	41,538	132,574	159,109	161,107	200,647
Barley**	14,691	41,792	226,059	135,626	240,750	177,418
Pineapple	-	-	9,924	175,913	9,924	175,913
Dry Chile	765	8,740	23,619	144,757	24,384	153,497
Dry Alfalfa	31,989	137,819	-	-	31,989	137,819
Soybeans	42,601	122,771	11,642	12,635	54,243	135,346
Cantaloupe	7,167	68,689	8,567	65,272	15,734	133,961
Peanuts	5,452	16,473	57,229	103,982	62,681	120,455
Squash	-	-	16,150	110,279	16,150	110,279
Onions	1,725	12,306	15,281	72,014	17,006	84,320
Oats	963	1,993	74,457	54,131	75,420	56,124
Lima Beans	1,307	2,283	46,393	51,422	47,700	53,705
Garlic	602	5,430	5,231	30,305	5,833	35,735
Flaxseed	4,369	10,917	14,133	22,755	18,502	33,672

Sources: Ministry of Water Resources, Directory of Irrigation Districts; and Ministry of Agriculture and Livestock, Directory of Agricultural Economics.

* Excluding cucumbers and cottonseed

** Including two forms of barley, that which is harvested whole and grain barley.

The cropping activities in the model also constitute factor demand activities. Factors are supplied by a separate set of factor supply activities, and there are balance equations which require equilibrium on the factor markets. The factor supply functions range from the perfectly elastic, for those inputs for which the agriculture sector is a price taker (e.g. chemicals, capital), to the perfectly inelastic (e.g., some categories of land). In the former category, factor prices are exogenous to the model, in the latter they are endogenous, and in intermediate cases they are endogenous within limits. Labor falls in the intermediate category. When factor prices are exogenous, the factor is regarded as a national resource, i.e., it has alternative uses in other sectors or in international trade. At the other extreme, factors in inelastic supply are purely sectoral resources which have no economic use outside the sector in the short-run; agricultural land and water are placed in this category.

Demands for land, labor, and water are defined at seasonal intervals.* All other inputs are treated on an annual basis, including services of farm machinery and draft animals. This is one of the instances of the model's being tailored to specifically Mexican conditions. Virtually all farm machinery (mainly tractors) is used in the irrigated areas of the central plateau and the arid northern zones. Due to the nearly uniform year-round climate in these areas, there is not a very pronounced degree of seasonality in aggregate demand for machinery services. Hence, to simplify an already complex model, the seasonal specification has been dropped in the case of machinery.

---

* Monthly in CHAC and four-monthly in the aggregated version.

Labor* is divided into three classes: farm owners plus their family labor, hired (landless) agricultural labor, and machinery operators. Local and interregional migration is permitted in the model for landless labor and for farmers on rainfed farms. Machinery operators constitute less than five percent of the agricultural labor force, and they do not appear to be a binding resource in Mexican agriculture, so they are assumed to be supplied in fixed proportion to machinery services - with infinite elasticity of supply. The wage for machinery operators is higher than the wage for hired labor, and both types of wages vary among regions, in accordance with observed behavior.

In any particular month, farmers are assumed to be willing to work for an own wage, or reservation price, which is lower than the hired labor wage. Thus the basic steps on the labor supply function are the following: (1) using the labor of the farmer and his family, (2) hiring local landless labor, (3) hiring surplus landless labor from other regions, and (4) hiring landless labor away from lower-productivity employment in other regions.** The model is structured in a form that permits ready adjustment of all wages, so that various experiments - such as measuring capital labor substitution - may be conducted.

4. Spatial Disaggregation

On the product supply side, each of the twenty submodels represents either irrigated, rainfed, or tropical cultivation, and each covers a particular set of counties or districts, which are not necessarily contiguous. In the case of rainfed and tropical agriculture, the sub-

---

* Labor in the model is discussed more fully in sections 6 and 7 below.

** This last step of course contains multiple steps.

models are defined on the basis of annual rainfall and altitude, which determine climatic conditions. Cropping and investment activities are specified by submodel. The submodels are grouped into four geographical regions, and labor constraints are specified for each region, in order to capture the differential labor mobility and wage rates which exist in Mexico. The regions are as follows (see map):

(I) The Northwest - an arid region of large scale irrigation along a thousand-mile coastal strip between the Gulf of California and the Sierra Madre Occidental, plus Baja California. Agriculture is more extensively mechanized here than in any other region.

(II) The North - the rest of the northern part of the country; this region is also extremely arid and cultivable only with irrigation except for the eastern portions near the Gulf of Mexico.

(III) The Central Plateau - an area of mixed rainfed and irrigated farms, concentrated along the course of the Lerma River; the farms are generally smaller than in the North and Northwest; twenty years ago this was the most productive region in Mexican agriculture, but it has been surpassed by the northern regions.

(IV) The South - tropical agriculture with very few systems of water control; due to the mountainous terrain, this region is the most remote from the major urban markets.

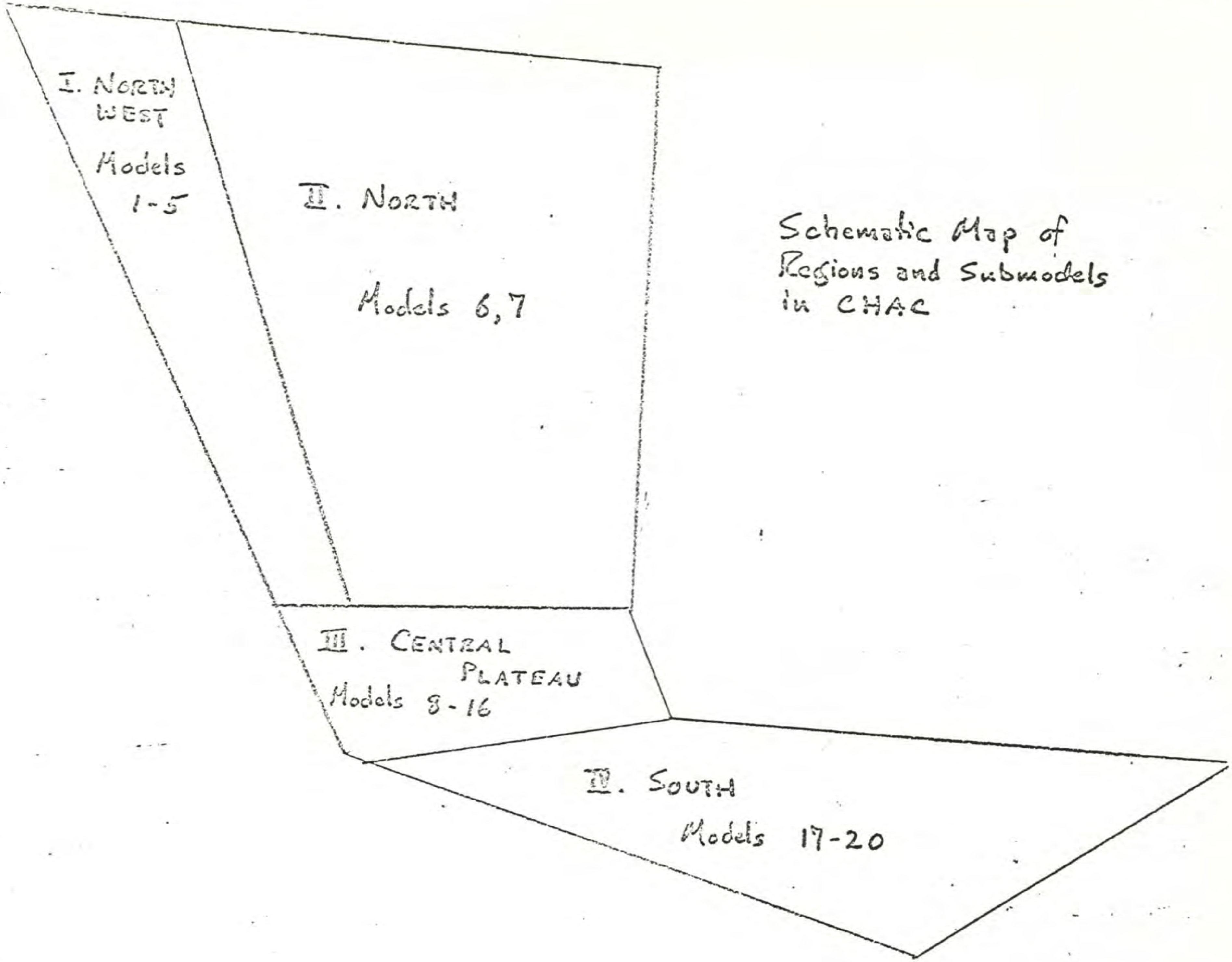
While there are landless agricultural laborers who live in each region, gaining a livelihood from part-time work on irrigated farms, the bulk of them reside in the Central Plateau region, where there is closer access to the major urban centers for part-time work and where small rainfed plots may be cultivated. The dominant direction of seasonal labor migration is between the Central Plateau and the North and Northwest. There also is some movement from the South to the Central Plateau and the northern regions, but due to the distances involved this is more apt to be permanent rather than seasonal migration. To help limit the size of the model, seasonal and permanent migration activities have been specified only for the directions of significant net flow, i.e. (a) from Central Plateau to Northwest, (b) from Central Plateau to North, and (c) from South to Central Plateau. Observed wages for hired labor are lowest in the South and highest in the Northwest. This reflects, at least in part, the relative abundance of labor in the tropical areas and the Central Plateau: migration is a gradual process and disequilibria in regional labor markets often persist for decades.

The local constraints for each submodel - primarily the annual and monthly bounds on land, water, and farmers - form a block in the block-diagonal production tableau. Since the constraints in one block are independent of all other constraints, additional submodels may be added to the system, with appropriate modifications in the coverage of the existing submodels. In this way, the model may be focused on the detailed choices in one geographical area, while treating other areas in a more aggregate fashion.

Demand functions are not specified for each submodel,* but rather nationally, except for a few food crops for which separate regional markets are introduced in the South and in the Northwest, due to the high cost of transportation between the tropics and other parts of Mexico. However, it is not assumed that each submodel can equally well supply the "national" market. Spatial price differential parameters are used which reflect the differential transport costs faced by each submodel area, based on the historical patterns of transportation. Thus the Northwest region farmers receive a lower farm-gate price for vegetables than the Central Plateau farmers do, for the latter are located closer to the major urban markets of Mexico City and Guadalajara. For export crops, proximity to major ports determines the spatial pattern of price differentials.

---

*For two reasons: there is insufficient information on the spatial distribution of demand, and the introduction of local demand activities would make the model much larger.



Schematic Map of Regions and Submodels in CHAC

Table 10-2 Spatial Components of CHAC

Region	Location 1/	Farm Type 2/	Submodels Number	Submodels Name
Northwest	Río Yaqui	I	1	Río Yaqui
	Culiacán, Rio Humaya, San Lorenzo	I	2	Culmaya
	Río Colorado	I	3	Río Colorado
	Comisión del Fuerte	I	4	El Fuerte
	3/	I	5	Residual Northwest
North	4/	I	6	North Central
	5/	I	7	Northeast
Central Plateau 18/	6/	LR	8	El Bajío A 7/
	6/	SR	9	El Bajío B 7/
	Alto Río Lerma, La Begoña	I	10	El Bajío Irrigated 7/
	Lands between 1,500 and 200 meters of elevation with 600 to 800 mm. of annual rainfall 8/	R	11	Temporal A
	Lands of 1,500 to 2,000 meters elevation with more than 800 mm. of rainfall 9/	R	12	Temporal B
	Lands of 1,000 to 2,700 meters elevation with 400 60 600 mm. of rainfall 10/	R	13	Temporal C
	Lands of 2,000 to 2,700 meters elevation with 600 to 800 mm of rainfall 11/	R	14	Temporal B
	Lands of 500 to 1,500 meters elevation with 700 to 900 mm. of rainfall 12/	R	15	Temporal E
	13/	R	16	Central Irrigated
South 18/	Lands of 0 to 500 meters elevation with 900 to 1,500 mm. of rainfall 14/	T	17	Tropical A
	Lands of 0 to 500 meters elevation with more than 1,500 mm. of rainfall 15/	T	18	Tropical B
	Lands of 500 to 1,500 meters elevation with more than 900 mm. of rainfall 16/	T	19	Tropical C
	17/	I	20	South Irrigate

Notes to table 10-2

1. For irrigation submodels, the location is defined in terms of the administrative irrigation districts of the Ministry of Water Resources (S.R.H). For rainfed and tropical areas, altitude and rainfall define the submodels, and each submodel's precise coverage is stated in terms of municipios (counties). Each municipio is assigned wholly to one submodel.

2. The farm types are as follows:
   - I - irrigated
   - LR - rainfed, large farms (ten has. or more)
   - SR - rainfed, small farms (less than ten has.)
   - R - rainfed
   - T - tropical

   In many of the irrigation submodels there are additional distinctions among farms, based primarily on efficiency in water use.

3. The remaining S.R.H. districts in the states of Baja California, Sonora, and Sinaloa: Santo Domingo; Guasave; Mocorito; Colonias Yaquis; Costa de Hermosillo; Río Altar; Río Altar, Pitiquito y Caborca; Río Mayo; Guaymas.

4. The nine S.R.H. districts in the states of Chihuahua, Coahuila, and Durango (including Don Martín, which is in both Coahuila and Nuevo León).

5. The nine S.R.H. districts in the states of Nuevo León and Tamaulipas.

6. The rainfed portions of the 17 municipios in Guanjuato which are at least partly contained in the S.R.H. districts of Alto Río Lerma and La Begoña.

7. In order to evaluate a set of investment alternatives which includes transforming rainfed land into irrigated land, submodels 8, 9, and 10 are solved together, known collectively as "El Bajío".

8. Mostly the states of Puebla, Guanajuato, Hidalgo, and Querétaro.

9. Mostly the states of Jalisco, Michoacán and Morelos.

10. Mostly the states of the northern part of the Central Plateau plus those further north.

11. Mostly the state of México.

12. Mostly the states of Oaxaca, Guerrero, Nayarit and Veracruz.

13. The 73 irrigation districts in the states of Jalisco, Mexico, Michoacán, Morelos, Hidalgo, Aguascalientes, Puebla, Querétaro, Tlaxcala, and Zacatecas. Virtually all of these districts are quite small compared to those of the North and the Northwest.

14. Mostly the states of Chiapas, Guerrero, and Veracruz.

15. Mostly the states of Tabasco, Campeche, and Yucatán.

10

Notes to table 2, cont.

16. Mostly part of Puebla, Chiapas, and Veracruz.

17. The 31 irrigation districts in the tropical states of Veracruz, Chiapas, Campeche, Yucatan, Guerrero, Oaxaca, Colima, and Nayarit.

18. The coverage of the various rainfed and tropical submodels does not fall entirely in the Central Plateau and Tropical regions, respectively. However, the regional designations are good approximations, and the wages in rainfed and tropical areas are very similar in magnitude to those of the Central Plateau and South regions in general.

In the case of the South, the submodels there may sell beans, for example, against a downward sloping local demand schedule and, by incurring a fixed transportation charge per ton, offer any additional maize on the national market. Similarly, the Central Plateau has the option of selling part of its output on the South's market, provided it incurs the transportation differential.

5. The Production Technology Set

CHAC contains 2,345 cropping activities to describe alternative techniques for producing the thirty-two crops. The range of variations in their activities is described fully in the following chapter; they are merely summarized here. Each cropping activity defines a yield and fixed proportions of the following current inputs: land (monthly), water (monthly, annual), labor (monthly), machinery services, draft animals, chemicals, purchased seeds, and short-term institutional credit. Relations between inputs and outputs are taken to be those which are observed (and projected) in each locale, and not necessarily the biological or profit-maximizing optima. In principle, possibilities for movement toward more nearly optimal input mixes could be represented with activities for extension services, but existing data do not provide a basis for doing so.*

The ratio of each input to output varies over the submodels for every crop. Some localities have shorter growing seasons than others, so the number of months of land differs; fertilization practices vary, especially between irrigated and non-irrigated areas; and for irrigation the amount of gross water release required at the dam depends on the length and condition of the canals, and this too varies from area to area.

---

* Work is underway for estimating water response and fertilizer response curves for future versions of CHAC.

In addition to these variations across submodels, there are systematic variations within many of the submodels in the input-output ratios, particularly in the amounts of water, machinery services, and labor per unit of output.*

For some of the irrigation submodels, the land is grouped into four classes, based on efficiency of gross water use.** For all of the submodels, alternative degrees of mechanization have been specified in CHAC: totally non-mechanized (all power operations done with draft animals) partially mechanized, and fully mechanized (no draft animal use). Obviously there can be many degrees of partial mechanization, but in actuality the choices are discrete and few, e.g., the plowing is done with either mules or tractors but not with both. To avoid overstating farmers' short-run flexibility with respect to technique, one-degree changes of technique are permitted but not two-degree changes. If the farmers in the area covered by one submodel use totally non-mechanized techniques, that submodel contains non-mechanized and partially mechanized techniques only. Similarly, it is assumed that fully mechanized farms may revert to partial mechanization but not to non-mechanization, under significant relative price changes.

The major advantage of mechanization, vs. use of draft animals, lies in land savings. A crop can be harvested, and the ground prepared for a new crop, significantly faster with tractors than with draft animals.*** In some cases, this time saving makes the difference of being able

---

* Variations in water-output ratios occur within five of the ten irrigation submodels, and the machinery and labor requirements per unit of output vary within all twenty submodels.

** Which in turn is due to terrain conditions, distance from the water source, and state of repair of the canals. The land classes are designated by the Secretaría de Recursos Hidráulicos.

***One might think that the same savings could be achieved by simply using more mule teams, but, as anyone who has worked with mules knows, there are limits to the number which one farmer can supervise.

to plant a second crop during the year. This saving is shown in the model by requiring fewer months of land with the more mechanized techniques, e.g., the first month of land is represented by a coefficient of 0.3 instead of 1.0 (one hectare for ten days instead of thirty days).

Differential land (and labor) requirements also constitute the distinction between two forms of the crop in the case of alfalfa: the crop may be sold green, at a lower price per ton, or left on the land longer and sold dry, at a higher price. In the case of barley, the farmer also faces a choice - of harvesting the entire plant and selling it as forage, or of using substantially less labor and harvesting only the grain. As in the case of alfalfa, there is a separate demand function for both types,* so prices move in the model in response to these production choices. For grain barley, there is an additional component on the demand side, the demand for malt grain. There is a minor amount of post-harvest on-farm processing for the grain which is destined for malt, but this is ignored as an approximation, so there are two domestic markets specified for grain barley: malt and non-malt.

For cotton also CHAC contains two markets, but this arises from the joint-product nature of the cotton crop. Separate demand functions are specified for both cotton fibre and cotton seeds, and in the case of seeds, the price depends partly on the volume of production of other oilseeds. Hence in the model, the profitability of growing cotton depends on (a) the demand schedule for cotton fibre, (b) the demand schedule for oilseeds, (c) the production surface for competing oilseed crops, and (d) the production surface for cotton.

---

*This is an over-simplified account. In fact, the two forms of alfalfa are contained in the same product group.

6. Factor Supply Activities

Three classes of factors may be distinguished in CHAC: those supplied at the level of each submodel, those supplied at the regional level, and those supplied at the sector-wide level. At the submodel level are supplied land, water, and the labor of farmers plus their families. Agricultural land is not priced, for it has no opportunity cost in the short-run, but the dual solution of CHAC yields the value of rents which accrue to the land. Similarly, endowments of water are not priced but the cost of tapping the water supply and providing it to farms is charged against the objective function. However, there is a price charged for the labor of farmers and their families; farmers may be fixed on the land in the short-run but their presence is due to a longer-run decision which is based in part on recognition of their opportunity cost. If it were assumed that farmers were willing to work for zero wages, cropping activities would enter the optimal basis which would not enter under more realistic assumptions, and hence all of the supply functions in the model would be biased toward overestimation of the supply offered at a given set of product prices. Also, unless extensive fiscal redistributional schemes are to be considered, policy oriented models must, if they are to provide solutions amenable to implementation, be based on wage assumptions not very different from actual wages.

Factors supplied at the regional level include hired labor and chemical inputs, and services of draft animals. This permits eventual inclusion of interregional price variations for delivered agricultural chemicals which follow from the transportation costs inherent in the spatial pattern of their production and use.* Sector-wide factor supply activities in CHAC

---

* This is not done in the version of CHAC used for the results reported here.

include those for credit, seeds, and machinery services. A sector-wide water pricing activity has been included in order to perform policy experiments regarding the effects of systematic sector-wide variations in water charges.

Most of the factor supply activities are straightforward; except for labor, all regional and sectoral inputs are assumed to be supplied with infinite elasticity.*

A schematic tableau for the entire matrix of coefficients is presented in figure 10-1. In this schematic rendering, there are two regions and five districts, instead of the four regions and twenty districts actually contained in CHAC. The cross-hatched areas represent blocks of zero coefficients; for non-zero coefficients the sign is indicated. An algebraic statement of the model, along with a listing of rows and columns, is found in section 10 below.

7. More on Labor

Labor activities and constraints constitute the most complex part of the factor supply set. One of the major purposes of the agricultural sector study is to measure the impact on employment patterns of various policies, and the labor components of CHAC have been designed accordingly. Some of the elements of the labor structure have been mentioned: monthly labor demands are generated within each submodel and these demands are met either with local labor or through inter-regional migration. Regional wage differentials are incorporated, and provision is made for a reservation price for farmers' own labor which is different than the wage for landless, or day, labor.

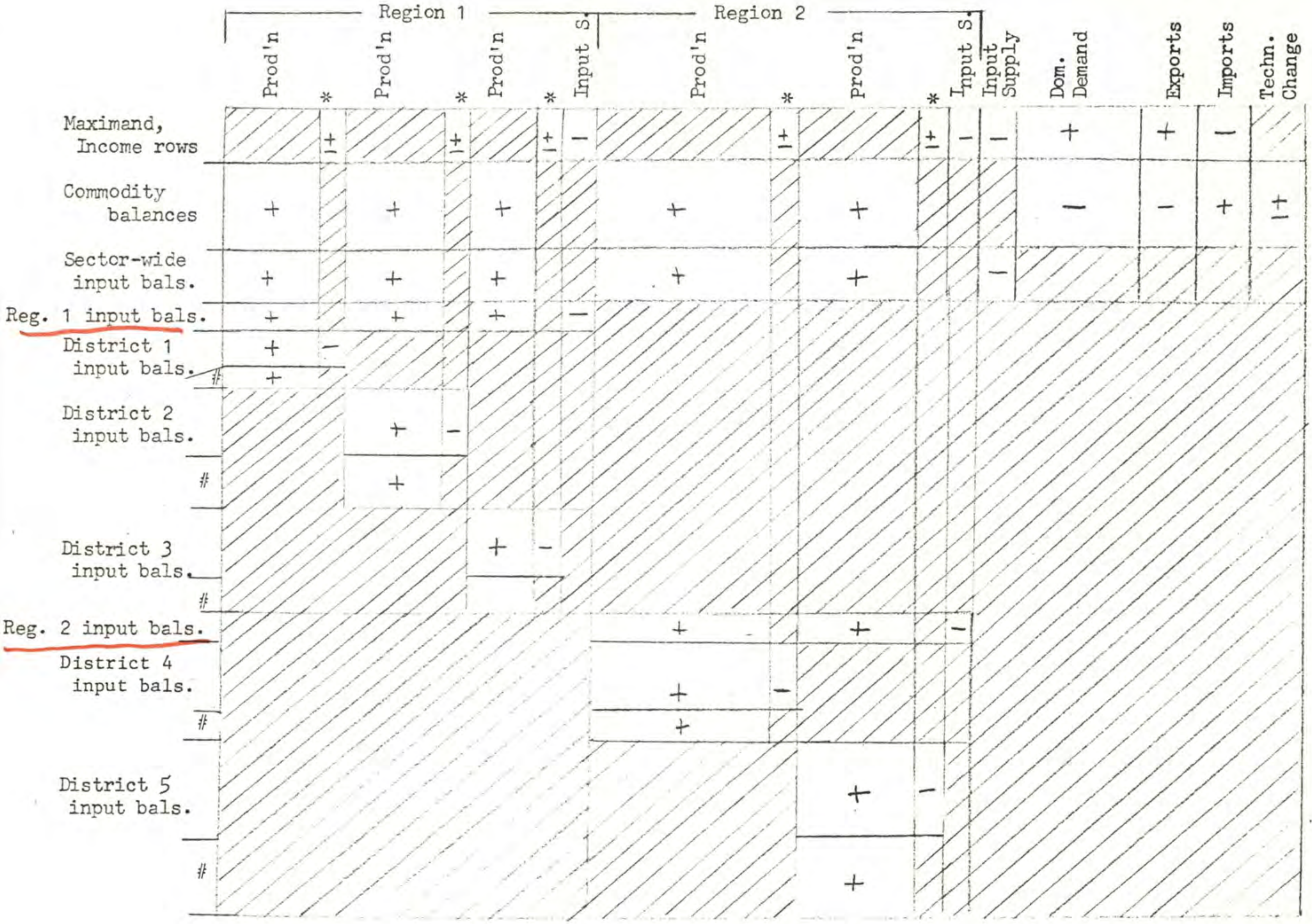
The number of farmers is fixed for each district, and the number of landless laborers is given for each region, i.e. rural-urban migration is either exogenous or determined through links with DINAMICO.** While farmers do migrate to cities, the number of farms in Mexico does not change very rapidly over time, so in the short-run the assumption that the number of farmers in each locality is given appears tenable.

---

* This assumption can be changed readily if conditions warrant.

** See chapter 14 for a discussion of rural-urban migration estimates resulting from the CHAC-DINAMICO linkage.

Figure 10-1 Schematic Tableau of CHAC



*District-level investment activities and activities for registering spatial price differentials.

#District resource constraints (land, water, farmer and family labor)

Farmers in non-irrigated areas in Mexico often work seasonally on irrigated farms, so this kind of labor transfer is allowed in the model, with the exception that people leaving tropical areas are assumed to move permanently rather than temporarily, since the distances are so great. The reverse flow, of farmers with irrigated land working on non-irrigated farms, virtually never occurs in Mexico, so it is not an option in the model.

The landless labor force is divided into four regional pools in the model, and if one region employs all the members of its pool in a particular month, it may draw redundant laborers from another region.

Regional wage differentials are incorporated in CHAC by multiplicative factors, so that the proportional differences remain constant when experiments are conducted with different base wage rates. Official "minimum" wage rates exist, but generally they are not enforced, so they have been used as the maximum wages in parametric variations on the price of labor and capital. The regional averages of official minimum wages are, in 1968 pesos/day, 19.5, 20.5, 24.0, and 26.0 for the South, Central Plateau, North and Northwest regions, respectively. In terms of the model structure, the South's wage rate is the base wage; solutions have been conducted with base wages of 13.5 and 16.5 pesos as well as 19.5 pesos,* maintaining the same proportional regional differences.

The model is structured so that any ratio of the farmers' own-wage to the day labor wage may be employed; in the solutions reported here it is assumed that the ratio is 0.5. This gives an own-wage for farmers ranging from 7.8 to 13.0 pesos/day, depending upon the region and

---

* At full employment, the daily wages of 13.5, 16.5, and 19.5 correspond to annual wages of 3,564, 4,356, and 5,148 pesos, respectively. In the 1960 prices of DINAMICO, these would be annual wages of 2,742, 3,351, and 3,960 pesos, respectively.

the base wage assumption, but recall that farmers' actual returns are much higher in many months.*

The submodels (district models) essentially reflect "representative farms" in each district, since the production structures are taken from average data for the district or part-district. Hence, even within a fairly disaggregated model there is a considerable degree of aggregation over farms; and one consequence is an overstatement of resource mobility within the district. For example, since reservoir water is allocated administratively, it may be reasonable to assume that it can be re-allocated in any manner, but the same cannot be said for the water from private wells. In labor, too, there is an overstatement of mobility: in CHAC the stock of farmers implicitly may be allocated in any manner among the farms in the district. In actuality, some farmers in a district may hire day labor in months when other farmers are idle. Farmers with irrigated land rarely work as seasonal laborers for other farmers, i.e. the low reservation wage applies only to work on their own farms.

To overcome CHAC's bias toward intra-district labor mobility, it has been specified so that farmers with irrigated land may not offer their labor services on a monthly basis, but only on a quarterly basis, while day labor is available monthly.** If both types of labor were supplied on a monthly basis, the lower reservation price of farmers would imply that day labor is hired only in themonths when all farmers in the district were fully employed. With the quarterly contract device, this is not the case.

---

* In typical solutions, the ratio of farmers' total income to wage income is about three.

** In the aggregated version, CHAQUITA, the same effect is achieved with four-monthly hire periods for day labor and annual periods for farmers.

For examples with a day labor wage of twenty pesos and a farmer reservation price of ten pesos, one-month peaks in labor demand will be met with hired labor but two-month peaks will be met with farmers on quarterly contracts.

Figure 10-2 illustrates the impact of the quarterly contract assumption on labor hire patterns. Of course, the quantity of labor demanded depends in part on the labor supply specification, but if it is assumed for the moment that the seasonal demand for labor is fixed, it might look something like the heavy line in the figure. If the reservation price is half the day labor wage, and if quarterly contracts are used for farmers, then day labor will be hired to meet the peak demands represented by shaded areas. Farmers will satisfy the remaining labor requirements; the dotted areas show the number of "farmer-days" for which the model is charged when in fact farmers are idle.*

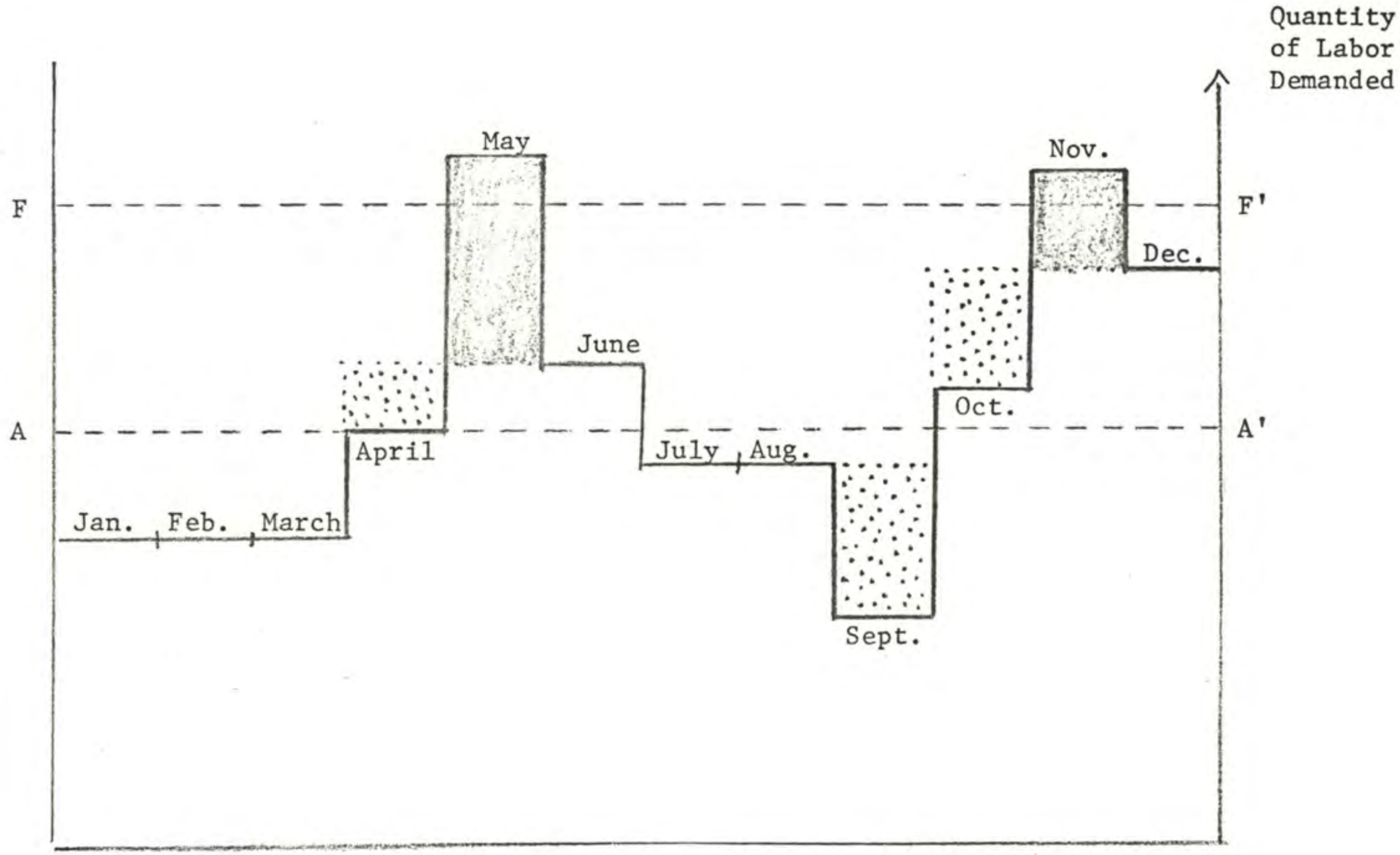
If farmer availability were specified in the form of annual contracts, then farmer hire would correspond to the number of man-days which lie below both the heavy line and the line AA'.Day labor hire would meet remaining requirements. And if farmers were available monthly, then all labor requirements up to the line FF', representing the total number of farmers in the district, would be met with farmers.

To summarize, the amount of labor hired in the model depends directly on four variables: (a) the wage rate for day labor, (b) the productivity of labor and other factors in the various cropping activities, (c) the ratio of the farmer reservation wage to the day labor wage, and (d) the length of the farmer contract. The latter two variables are subject to various assumptions, but they are interrelated. Whatever set of

---

* These might be thought of as the "fencing-mending" periods.

Figure 10-2. Alternative Labor Hire Patterns



assumptions is adopted, it should be designed to offset the implicit assumption of complete farmer mobility within a district.

The model's employment results do not appear to be particularly sensitive to the level of the reservation price. An experiment with one of the submodels (Río Colorado), solved in isolation, yielded the following results:

	Solution No. 1	Solution No. 2
(1) Wage ratio*	.62	.23
(2) Total employment**	1448	1484
(3) Farmer employment***	1280	1331
(4) (3)/(2)	.884	.897
(5) Non-farmer man-years (thousands)	6.4	5.8

Apart from the demands for labor generated in the district-level cropping activities, the labor rows and columns may be considered as a separate submatrix of the model. The accompany schematic tableau (figure 10-3) displays the elements of the labor submatrix, and the rows and columns are defined in table 10-3.

---

* Ratio of the farmer reservation price to the wage for day labor.

** In 10,000 man-days, including machinery operators (about 4% of the total).

*** In 10,000 man-days.

Figure 10-3

AGRICULTURAL SECTOR LABOR MATRIX

Row \ Column	SALS	SAL1	SAL2	SAL3	SAL4	RES1	RES2	RES3	RES4	LMAN	LMANt	dDLt (I)	dDLt (II)	dDLt (IIIi)	dDLt (IIId)	dDLt (IVi)	dDLt (IVt)	dFLg (I)	dFLg (II)	dFLg (IIIi)	(IIId) dFLt	(IVi) dFLg	(IVt) dFLt	Cropping Activities (I)	(II)	(IIIi)	(IIId)	(IVi)	(IVt)	MDL31t	MDL32t	MDL43A	MA33t	MA44A	RHS
SØB	-W																												ε⋯ε	ε⋯ε	12ε	δ⋯δ	12δ	(Max)
SIN1	-W																												ε⋯ε	ε⋯ε	12ε	δ⋯δ	12δ	≥ Y_1
SIN2	-W					a_1	a_2	a_3	a_4																				ε⋯ε	ε⋯ε	12ε	δ⋯δ	12δ	≥ Y_2
SIN3																																			≥ Y_3
SSAL	–	w_1	w_2	w_3	w_4																														= 0
S1SAL		–				+k						+⋯+																							≤ 0
S2SAL			–				+k						+⋯+																						≤ 0
S3SAL				–				+k						+⋯+	+⋯+																				≤ 0
S4SAL					–				+k							+⋯+	+⋯+																		≤ 0
S1RES						–												3⋯3																	= 0
S2RES							–												3⋯3																= 0
S3RES								–												3⋯3	+⋯+														= 0
S4RES									–													3⋯3	+⋯+												= 0
SMAN										-12	+⋯+																								= 0
SMANt											-2.2⋱-2.2	+⋱+	+⋱+	+⋱+	+⋱+	+⋱+	+⋱+	‡⋱‡	‡⋱‡	‡⋱‡	+⋱+	‡⋱‡	+⋱+												= 0
S1MANt												+⋱+																		–⋱–					≤ L_1
S2MANt													+⋱+																		–⋱–				≤ L_2
S3MANt														+⋱+	+⋱+															+⋱+	+⋱+	–⋮–	–⋱–		≤ L_3
S4MANt																+⋱+	+⋱+															+⋮+		–⋮–	≤ L_4
S3MIGt																														+⋱+	+⋱+	–⋮–	–⋱–		≤ M_{3t}
S3MIG																														+⋯+	+⋯+	-12	–⋯–		≤ M_3
S4MIG																																+12		+12	≤ M_4
I) dMØNCt												–⋱–						≡⋱≡						+⋮+											≤ 0
II) dMØNCt													–⋱–						≡⋱≡						+⋮+										≤ 0
IIIi) dMØNCt														–⋱–						≡⋱≡						+⋮+									≤ 0
IIId) dMØNCt															–⋱–						–⋱–						+⋮+								≤ 0
IVi) dMØNCt																–⋱–						≡⋱≡						+⋮+							≤ 0
IVt) dMØNCt																	–⋱–						–⋱–						+⋮+						≤ 0
I) dAGRIq																		++++																	≤ A_d
II) dAGRIq																			++++																≤ A_d
IIIi) dAGRIq																				++++															≤ A_d
IIId) dAGRIt																					+⋱+												+⋱+		≤ A_d
IVi) dAGRIq																						++++													≤ A_d
IVt) dAGRIt																							+⋱+											+⋮+	≤ A_d
IIId) dMIGRA																																	+⋯+		≥ MA_3
IVt) dMIGRA																																		+12	≤ MA_4

$a_i = k w_i W$

Table 10-3 Rows and Columns in the Labor Matrix

Row Symbol*	Purpose
SOB	Objective function
SIN1-SIN3	Measures of sectoral income
SSAL	Sectoral wage accounting equations
SrSAL	Regional wage accounting equations
SrRES	Reservation price accounting equations
SMAN	Total annual labor accounting equations
SMANt	Total monthly labor accounting equations
SrMANt	Regional monthly constraints on day labor
S3MIGt	Monthly constraints on migration of day labor out of region 3 (Central Plateau)
SrMIG	Annual constraints on migration of day labor out of South and Center regions (r = 3,4)
dMONCt	Submodel labor balances
dAGRIq	Quarterly submodel constraints on the supply of farm labor, for irrigation districts only.
dAGRIt	Monthly submodel constraints on the supply of farm labor, for non-irrigation submodels
dMIGRA	Annual constraints on migration of farmers, non-irrigation submodels in regions 3 and 4 only.

* The letter r (= 1, ..., 4) is the regional index;
t (= 1, ..., 12) is the monthly index;
q (= 1, ..., 4) is the quarterly index;
d (takes on alphabetic values) is the submodels index.

(Continued..)

Table 10-3 (cont'd...)

Column Symbol	Purpose
SALS	Sectoral wage charging activity
SALr	Activities for charging regional wage differentials
LMAN	Sector annual employment counter(man-years at full employment equivalent)
LMANt	Sector monthly employment counter
dDLt	Monthly day labor supply activities in each submodel
dFLq, dFLt	Farm labor supply activities by submodel, quarterly in irrigation submodels, monthly otherwise.
MDLrr't	Migration activities for day labor from region r to region r' by months (rr' + 31, 32, 43).
MA33t	Migration activities in region 3 for farmers to the pool of day laborers.
MA44A	Migration activities for region 4 farmers on annual basis.

The remainder of this section constitutes a "guided tour" of the tableau, to clarify its driving mechanisms.* The monthly demand for labor is generated by cropping activities in the submodels. This demand is registered in the submodel-level labor balance equations (dMONCt). To meet the demand are a set of activities supplying labor, a set of activities charging the objective function for labor use, and a set allowing migration of labor. Labor can be supplied by local farmers, or by drawing upon the regional pool of landless laborers. In the r^{th} region, the cost to the model of a unit of farmer labor working on his own farm is kw_rW, where W is the base wage rate, w_r is the proportional regional wage differential,

---

* In the tableau, a "+" signifies + 1, and a "-" signifies -1.

and k is the ratio of the farmer reservation price to the wage. For non-irrigated farms, on which the incidence of labor demands is markedly seasonal, there are monthly constraints on the number of farmers (dAGRIt), and for irrigated submodels, quarterly constraints (dAGRIq), following the quarterly contract approach discussed above.

The activity SALS charges wage costs, and the four activities SALr, r=1, 2, 3, 4, apply the regional wage differentials. Activities RESr channel farmer employment into the wage charging activity, and at the same time they apply the ratio of the reservation wage to the day labor wage. Total employment, in man-years, is counted by the activity LMAN, and monthly employment is counted by LMANt.

Day labor is supplied to the submodels by a set of activities (dDLt) on a monthly contract, at a wage rate of $w_r W$, drawing upon four regional pools of day laborers, defined by the set of monthly constraints (SrMANt). The sizes of the regional pools are given by the magnitudes L_i on the right-hand side of the regional labor balances. The regional pools may be augmented through use of the migration activities. There is a seasonal movement of labor, following the seasonal demands associated with the cropping cycle. The model distinguishes between migration of day labor and of farmers. Migration flows of day labor on a monthly basis are allowed from region 3 (Central) to regions 1 (Northwest) and 2 (North) and on an annual basis from region 4 (South) to 3. Migration from, e.g., 3 to 1, increases the pool of day laborers in 1, reduces it correspondingly in 3, and is constrained both by a monthly and an annual limit (rows S3MIGt and S3MIG).

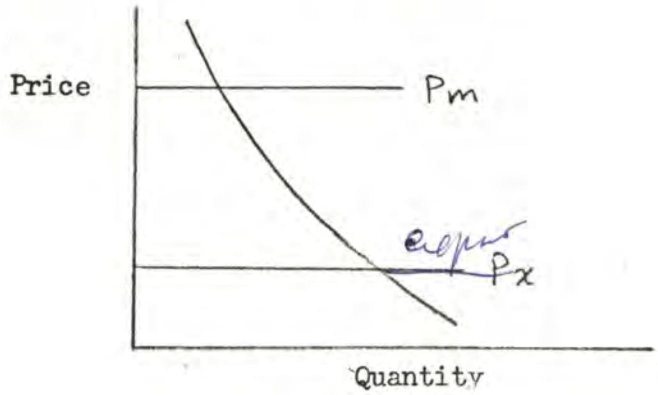
If labor is fully employed in one region in a given month and there is unemployment in another region in that month, labor will move between the regions to seek employment. If labor is fully employed in both regions in that month, it still will move from one region to another if its productivity differential in the two regions exceeds the wage differential. The numbers ϵ and δ in the migration activities are accounting devices; they are negative numbers arbitrarily small in magnitude, and $|\epsilon| < |\delta|$. They insure that, if there is surplus labor in both regions, there will be no interregional migration, and also that landless labor will migrate out of a region before farmers do. Except for the regional wage differentials, which partly reflect interregional transport costs, there is no explicit cost of migration in the model.

As noted earlier, day labor in the model may migrate from the Center to either of the northern regions, or from South to the Center. Given the present labor surplus in the Center, the latter movement occurs if seasonal migration northward is sufficiently strong to exhaust the Center's pool in a least one month. In that case, movement from the South to the Center would, by releasing constraints S3MIGt and S3MIG, permit more movement out of the Center northward, and hence allow a double migratory movement (at a cost of 2ϵ).

8. The Structure of Demand

The major departure in CHAC from the conventional structure of sector planning models is in the formulation of demand. In most sector planning models, the problem addressed is either that of minimizing the costs of producing a fixed bundle of output or of maximizing value added at exogenous product prices. The model ENERGETICOS, presented in chapter 17, is an example of the former. In CHAC product prices are endogenous. For a particular product, the demand function is illustrated in Figure 10-4.

Figure 10-4



In the diagram, p_m and p_x refer to import and export prices respectively. The difference between these reflects the difference between effective import and export prices, and it may be large for some bulky agricultural products. Export and import prices are treated as fixed and exogenous, although this assumption can readily be relaxed.* Also, for convenience of exposition, all demand functions are assumed to be linear. This assumption, too, can be relaxed.

The purpose of this treatment of demand is threefold. Firstly, it allows a model solution to correspond to a market equilibrium. The effects of various policies, e.g. subsidizing or taxing product prices or varying the exchange rate, etc., can then be investigated. Secondly, it allows the model greater flexibility. For instance, substitution between capital and labor, corresponding to different ratios of the wage rate to the rate of interest, can occur not only directly through the technology set or through changes in the commodity mix of output, but also through substitution in demand due to changing relative prices of outputs which are more or less labor or capital intensive. Thirdly, it enables a more realistic appraisal of the benefits (and particularly of their distribution as between producers

* For non-traded commodities, the demand function is specified between arbitrarily wide bounds which reflect the relevant range of potential prices and quantities.

and consumers) accruing from an increase in agricultural output. In the not-unlikely situation of agricultural production for the domestic market at prices between those at import and at export, and where domestic demand is price inelastic, then the financial return to producers as a whole from an increase in output is negative. For consumers, the benefits are positive.

Two forms of market equilibrium are distinguished. The first, the competitive case, involves producers acting as price takers and equating marginal costs to the prices of products. In the second, the monopolistic case, the sector maximizes its net income by equating marginal costs to the marginal revenue of products. As noted earlier, for agriculture the equilibrium prices and quantities of the competitive case correspond more closely to reality, but the monopolistic case proves to be useful for endogenous measurement of sector income.

The basic derivation of objective function arguments is as follows. For simplicity of exposition, it is first assumed that no external trade occurs. Import-export opportunities are introduced later. The set of domestic demand functions is written, assuming linearity, as

$$p = a + Bq \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

where p is a J x 1 vector of prices

a is a J x 1 vector of constants

B is a J x J matrix of demand coefficients

q is a J x 1 vector of quantities

Defining c as a J x 1 vector of marginal costs,* the objective function for the competitive case becomes

$$Z = q'\left[\,a + \tfrac{1}{2}\,Bq - c\,\right] \quad . \ . \quad (2)$$

which yields the equilibrium condition that

$$p = c \quad . \ . \ . \ . \ . \ . \quad (3)$$

---

* As is evident from previous sections of this paper, the supply functions of the model are in fact more complex than is implied by this simplified exposition.

The objective function, Z, can be decomposed into components which correspond to consumer surplus and producer surplus:

$$CS = \tfrac{1}{2}\, q' \left[a - p \right] = - \tfrac{1}{2}\, q'Bq \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4)$$

$$PS = q' \left[p - c \right] = q' \left[a + Bq - c \right] \; . \; . \; . \; . \; . \; . \; . \; . \; (5)$$

The appropriate objective function for the monopolist case is

$$Y = q' \left[a + Bq - c \right] . \; . \; . \; . \; . \; . \; . \; . \; . \; (6)$$

which yields the equilibrium conditions that

$$a + 2\, Bq = c \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (7)$$

where the left-hand term is a vector of marginal revenues.

In both cases, the maximand involves a quadratic form in q. For nonlinear demand functions, the maximand is nonlinear and nonquadratic. Problems arise in practice because,with existing computer codes, nonlinear programming models rapidly approach the limits of computer technology as the models become large. For this reason, approximation procedures were sought in order to take advantage of the computational efficiencies of linear programming. Two such procedures have been developed: the first for the case where estimates of the coefficients of the matrix B are available, and the second where less information is to be had on the structure of demand interrelationships. The first case is a problem in nonseparable programming, but the second has been defined to be separable. For the first, a new approximation technique in convex programming was developed,* and for the second the technique used is similar to the grid linearization of separable programming.**

---

* See Candler, Duloy, and Norton (1971).

** See Clair Miller (1963)

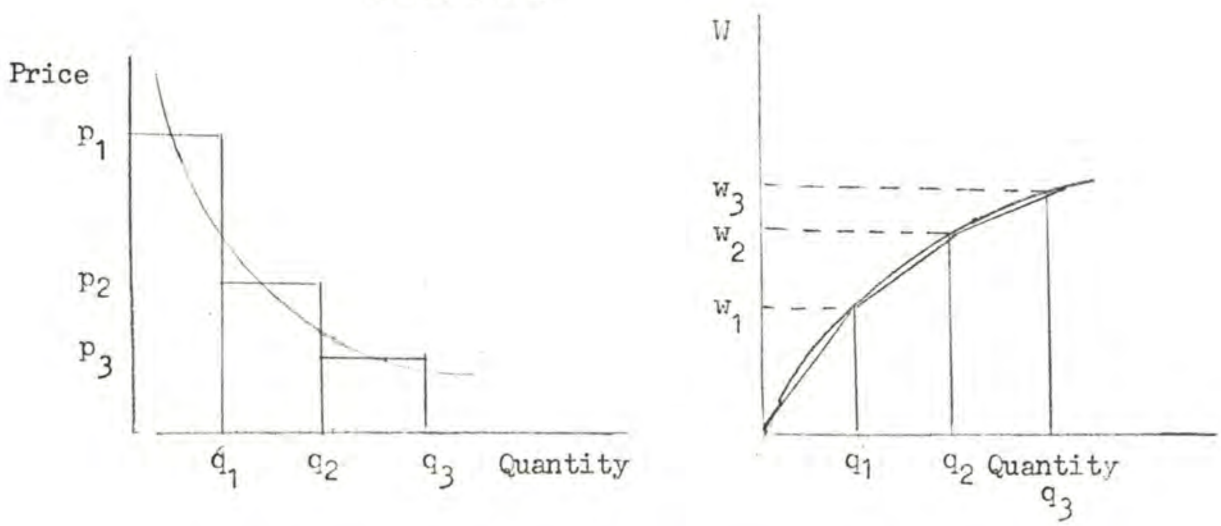
The first mention of the possibility of maximizing consumers' and producers' surpluses to achieve the competitive equilibrium in an optimizing model was made by Samuelson.* In models of spatial equilibrium with linear demand functions, the quadratic maximand (2) has been used directly by Takayama and Judge.**

In writing the demand structure of CHAC, recall that costs are accounted for in factor-supply activities, and hence the demand activities account only for the areas under the demand function (in the competitive case) or the area under the marginal revenue function (in the monopolist case). For one product, in the competitive case, this area is***

$$W = q' \left[a + \tfrac{1}{2} Bq \right] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

which is the function sketched in Figure 10-5 together with the corresponding demand function, assuming only three segments in the approximation.

Figure 10-5



The linear programming tableau corresponding to the segmented approximation of the function W for one product is the following:

---

* P.A. Samuelson (1952).

** See T. Takayama and G. Judge (1964, 1971).

*** For the monopolistic case, it is $R = q' \left[a + Bq \right]$, where R is the total revenue function.

	Cropping Activities			Selling Activities			RHS
Maximand	$-c_1$	$-c_2$	$-c_3$	w_1	w_2	$w_3 \cdots\cdots w_n$	(MAX)
Commodity Balance	y_1	y_2	y_3	$-q_1$	$-q_2$	$-q_3 \cdots\cdots -q_n$	≥ 0
Demand Constraint				1	1	$1 \cdots\cdots 1$,	≤ 1

where the c_i are costs associated with crop-producing activities,

the y_i are yields of the product,

the w_i are values of W corresponding to q_i, and

the q_i are the total quantities sold at the limit of each segment of the function W.

Notice that by the convexity of W, no more than two of the n selling activities appear in the optimal basis.

The main point of this formulation is that the demand function (or the area function W) can be approximated as closely as desired without adding additional rows to the linear program.

This approach is readily extended to two or more products. For products in which the utility function is separable, there is one commodity balance and one convex combination constraint per product. For the non-separable case, there is one commodity balance per product and one convex combination constraint for the entire set of selling activities. For n nonseparable commodities, effectively the W-function in n space is directly approximated by superimposition of an n-dimensional linear grid. For example, take the case of two commodities, the first of which is segmented into two parts, q_{11} and $q_{12}-q_{11}$, and the second of which is segmented into three parts, q_{21}, $q_{22}-q_{21}$, and $q_{23}-q_{22}$. The tableau for the grid linearization is as follows:

	Cropping Activities				Selling Activities						RHS
	Crop 1		Crop 2								
Maximand	$-c_{11}$	$-c_{12}$	$-c_{21}$	$-c_{22}$	w_{11}	w_{12}	w_{13}	w_{21}	w_{22}	w_{23}	(MAX)
Commodity Balance 1	y_{11}	y_{12}			$-q_{11}$	$-q_{11}$	$-q_{11}$	$-q_{12}$	$-q_{12}$	$-q_{12}$	≤ 0
Balance 2			y_{21}	y_{22}	$-q_{21}$	$-q_{22}$	$-q_{23}$	$-q_{21}$	$-q_{22}$	$-q_{23}$	≤ 0
Demand Constraint					1	1	1	1	1	1	≤ 1

where the c_{ij} are costs for the i^{th} crop in the j^{th} activity producing that crop,

the y_{ij} are yields of the i^{th} crop in the j^{th} activity producing that crop,

the q_{ij} define the commodity bundles of the two products, and

the w_{ij} are the values of W corresponding to the i^{th} level of the first commodity and the j^{th} level of the second.

The competitive and monopolistic cases can be combined by casting the total revenue function R as a constraint instead of objective function. It then becomes a constraint on producers' incomes at endogenous prices, and it corresponds to the policy instrument of supply controls.* The price-quantity equilibrium in the model will move away from the competitive point and toward the monopolistic point to the extent necessary to satisfy this constraint.

In the treatment above, it has been assumed that the off-diagonal elements of the matrix B were known or could be estimated. Frequently, this information is not available. For CHAC, the available information consisted of crude estimates of own-price elasticities for a number of individual

---

* Solutions reported in chapter 13 indicate that producers' incomes can be raised substantially with a relatively small loss of social welfare, involving mainly a transfer between consumers' and producers' welfare.

commodities and commodity groups. An approximation procedure developed for this situation of limited information has the following properties:*

(i) Because of the lack of information on the off-diagonal elements of B, the system does not reflect a complete range of price interdependence amongst commodities. Instead, it distinguishes between individual commodities with prices determined independently of other commodity prices** and groups within which there are fixed relative prices. The latter constitute a number of demand groups. For example, the various vegetable oils constitute one such demand group, in which the relative prices of soybean oil, peanut oil, safflower oil, etc. are held constant. Again on the demand side, the price for each commodity group is determined independently of the prices of other commodity groups or of individual commodities.

(ii) For any commodity group, both the consumers' surplus and producers' gross revenue are independent of the commodity mix in that group. Within a group, the marginal rate of substitution between any two products is constant and equal to the reciprocal of the ratio of their prices. Again, within a group, the derivative of the objective function with respect to the production of a particular product in the group, holding constant the production of all other products in the group, is equal to the price of that product.

---

* Proofs of these properties, plus an algebraic statement of the linear programming tableau, are found in Duloy and Norton (1971).

** This, of course, applies to the demand structure only. There is interdependence in product prices amongst all commodities arising from the interdependence of marginal costs on the supply side.

(iii) The system is structured such that substitution amongst products within a group is constrained within pre-assigned bounds on the commodity mix.

(iv) The system preserves the desirable property of the linearization of a quadratic form presented above, so that the area function, W, can be approximated to any desired degree of accuracy by adding activities without adding additional rows.

(v) The revenue function is similarly linearized, so that the demand activities have coefficients in rows defining producers' profits and incomes. These rows can be used as alternative objective functions, implying monopolist behavior of the sector, or used as constraints upon the social welfare maximization encompassed by the competitive equilibrium objective function. Similarly, this last can be included as a constraint upon monopoly behavior.

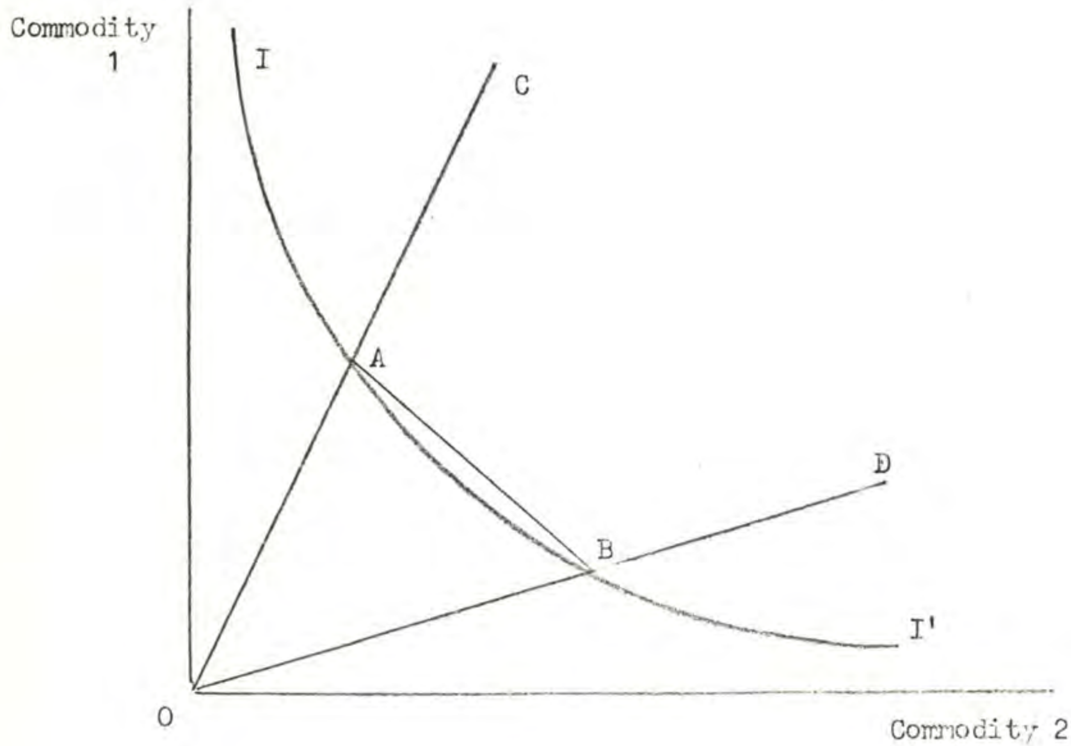
(vi) Export selling activities are included as additional demand activities for individual products, and import activities are added as alternatives to domestic producing activities.* Import activities will never enter the optimal basis under the monopolistic objective unless the model also includes a social welfare constraint.

The approximation is equivalent to linearizing the indifference surface between bounding hyperplanes. For a two-commodity group, Figure 10-5 provides an illustration.

---

* Such activities, of course, can also be included in the "full information" demand structure.

Figure 10-5



The rays OC and OD define limits on the commodity mix within the group, and the segment AB is the locus of mixes permitted in the model. Its slope, the marginal rate of substitution between the commodities, is set equal to the reciprocal of base-period relative prices. In the "full information" case, essentially the indifference curve II' is approximated in piece-wise linear fashion with as many segments as desired.

The price and income elasticities of demand actually used in the initial version of CHAC are listed in Table 10-4.

Table 10-4 Crops, Types of Cultivation, and Price and Income Elasticities of Demand in CHAC

Crop	Type of Cultivation: Irrigated	Temporal	Tropical	Income Elasticity	Price Elasticity
Wheat	*	*		.315	-.10
Maize	*	*	*	-.453	
Green Chile	*	*		-.119	-.20
Dry Chile		*		-.119	
Sugar Cane	*		*	.117	-.25
Beans	*	*	*	.330	-.30
Rice	*		*	.250	
Potatoes	*	*		.330	
Chickpeas	*	*		.330	
Tomatoes	*	*		.409	-.40
Onions	*			.598	-.20
Garlic	*			.598	
Cucumber	*			.598	-.60
Squash		*		.330	-.40
Lima Beans	*	*		.245	-.40
Forage maize	*	*	*	.500	-.30
Oats		*		.500	
Grain sorghum	*	*	*	.500	
Forage barley	*			.500	
Grain barley	*	*		.500	
Green alfalfa	*			.500	
Chickpeas	*	*		.500	

Table 10-4, cont'd...

	Type of Cultivation			Income	Price
	Irrigated	Temporal	Tropical	Elasticity	Elasticity
Malt barley	*	*		.460	-.10
Cotton fibre	*	*		.639	-.50
Cotton seed	*	*		.614	-1.20
Sesame	*	*	*	.614	
Flaxseed	*	*		.614	
Safflower	*	*		.614	
Soybeans	*		*	.614	
Peanuts	*	*		.614	
Peanuts	*	*		.330	-.20
Strawberries	*			.330	-2.00
Pineapple			*	.330	
Watermelon	*			.330	
Cantaloupe	*			.330	
Tobacco			*	.817	-.10

Notes to Table:

1. Some products appear in more than one demand group and hence there are multiple domestic markets for these products (maize, peanuts, chickpeas, and barley; malt barley and grain barley are the same product on the supply side).

2. In specifying the demand functions in the model, income elasticities are first applied to shift equilibrium points for individual products and then price-elastic demand functions are fixed on these new equilibrium points.

Sources:

Income elasticities - Projections of Agricultural Supply and Demand, 1965 - 1975, study forMexico conducted jointly by the Bank of Mexico, the Mexican Ministry of Agriculture and Livestock, and the U.S. Department of Agriculture (1965).

Price elasticities - unpublished studies by L. Barraza, J. Silos, and others.

9. Time and Investment Choices

CHAC is a static model. Its size makes it expensive to obtain simultaneous multi-period solutions, so a one-period statement has been adopted. It is, however, solved for different points in time with appropriate projections of exogenous data. Investment activities are included, and thus while the timing of investments cannot be addressed, the alternative investment projects can be ranked.

The model is based on data for 1968*, and solutions are conducted for 1968 and 1974. The base-period solutions were used to de-bug the model, and solutions for the latter year constitute the policy experiments. Since the investment projects in the model are small-scale in nature, the time lapse between the experiments (1971) and the solution date is sufficient for implementation of investment programs formulated with the assistance of the model. The absorptive capacity limitations which constrain the amount of investment in any one locality in any year cannot be identified easily, so it was decided to solve for the final year of a single three year period. This also conforms with DINAMICO, which treats three-year periods ending in 1971, 1974, etc.

Endowments of labor are projected from 1968 to the solution year. No attempt is made to estimate rural-urban migration within CHAC; the labor force is projected at the natural increase rate, and rural-urban migration is derived from the linked solution with DINAMICO. Export demand also is projected forward to the solution period, and disembodied technical progress is incorporated for purchased inputs and associated yield increases.

---

* 1967 - 69 for production, yields, and other variables subject to short-term fluctuations.

The major difference, in terms of effects on the solutions, between the 1968 and 1974 versions is the rightward shift over time of the domestic demand functions as a consequence of income growth. Income elasticities of demand for agricultural products are applied.*

The investment choices in CHAC are of the short-gestation lag variety, such as well digging, canal lining, and land levelling, which take less than a year. For these, the investment is assumed to take place instantaneously and the resulting benefits also are assumed to commence immediately. In the model, one year's (annualized) costs and benefits are entered in the objective function and income constraints. Annualized investment costs are defined to be one year's linearized depreciation (= amortization) and one year's interest charges against the full capital cost. Current operating costs of a project like a well are charged through current use activities. In the case of the three types of investment mentioned, the "output" of the investment activity is an additional unit of irrigated land, an additional unit of net water availability, or an additional unit of higher-yield irrigated land.

Investment choices of longer gestation lag may also be represented through annualized costs and benefits. While this treatment does violence to some aspects of project analysis, it permits assessment of all major forms of investment in a locality for purposes of measuring the marginal efficiency of capital schedule. (See chapter 12 on the project model.)

---

* See Table 10-4

10. Algebraic Statement and Tableaus

In the case of large-scale programming models, matrix tableaus often are as helpful as algebra in revealing the structure of the model. For this reason, tableaus have been presented above for the overall schema, for labor, and for demand, and some more are presented in this section for particular aspects of factor supplies and investment. Nevertheless, the algebra is useful,* so a statement is given here. A list of types of rows and columns is given also. For readers interested in undertaking similar exercises, there exists a two-volume set of "black books" on CHAC, which contain more than one hundred pages of tableaus, basic assumptions, and basic data, but not including computer listings or results. These black books have been the primary means of managing the technical aspects of a large-scale undertaking.**

For the algebraic statement, special notation has been devised. Capital letters represent vector variables or right-hand side values, small letters indexes, and Greek letters vector coefficients or scalars. Small Roman letters are used for pric In raw form, some of the vector symbols are burdened with several superscripts and subscripts, but in most equations only part of the vector is relevant, so an abbreviated notation is utilized. For example, the complete production vector is denoted X^{dz}_{hij}, but the subset which corresponds to all the vectors for producing crop j in district r is written X^{r}_{j}, and the total production of crop j in district r becomes $y^{r}_{j}X^{r}_{j}$, where y^{r}_{j} signifies the row vector of yields for those activities producing crop j in region r.

---

* Particularly for writing instructions for matrix generating computer routines.

**These may be consulted at the Development Research Center of the World Bank.

Given this convention, the set of symbols in full form is set out in table 10-5. The equations of the system may be written as follows:*

Table 10-5 Notation for Algebraic Statement**

Symbol***	Description	Superscripts, Subscripts
I. Variables		
X_{hij}^{dz}	Crop production	d = district z = zone (subdistrict) h = degree of mechanization i = type of irrigation j = crop n = class of investment
I_n^d	Fixed investment	
T_j^d	Inter-district price differences	
A^r	Regional supply of draft animal services	r = region
F^r	Regional supply of chemical inputs	
C	Sectoral supply of short-term credit	
K	Sectoral supply of machinery services (in tens of days)	
S	Sectoral supply of purchased seeds	
D_{ms}^g	Domestic demand	g = crop group m = commodity mix s = demand segment
E_j	Exports	
M_j	Imports	
P_j	Technical progress	

* The number of equations of each type is written in the left-hand column.

** The submodel for El Bajío, the project district, is more complex than this description implies. It is described fully in chapter 12.

*** For consistency within this presentation, the labor notation is adopted directly from the tableau on labor. See table 10-3 for names and definitions of labor variables.

Table 10-5, continued

K'	Sectoral supply of machinery services (in 10,000 pesos)
C^d	District-level counter for short-term credit
CP	Private long-term capital used
CT	Total long-term capital used

II. Parameters

α^g_{mj}	Quantity of crop j demanded in mix m of group g*
ω^g_s	Entry in maximand for demand group g and demand segment s (ie, weighted average price for segment s of all crops in the group)
ρ^g_s	Entry in income rows for demand group g and demand segment s (ie, weighted average marginal revenue for segment s of all crops in the group)
δ^g_m	Entry in the demand convex combination constraint for demand group g and mix m
	Ratio of region r day labor wage to region 4 day labor wage**
K	ratio for farmer reservation wage to day labor wage***
δ^{dz}_{ijt}	Water input coefficients (i=i' for gravity-fed water', i=i" for well water, t= month)
σ^d_j	Purchased seed input coefficients
ϕ^d_j	Chemical input coefficients

*The vector α_j is the union over g and m of all coefficients α^g_{mj}.

**This symbol is w_r in the labor matrix tableau of Figure 10-3.

***This symbol is k in the labor matrix tableau.

Table 10-5, continued

μ^{d}_{hj}	Machinery services input coefficients
β^{d}_{hjt}	Labor input requirements
θ^{d}_{hj}	Draft animal services input requirements
χ^{d}_{n}	Capital costs per unit of investment project (n = class of investment project)
τ^{d}_{hj}	Credit input requirements
III. Prices	
p^{e}_{j}	Exports
p^{m}_{j}	Imports
p^{ℓ}	Labor (Region 4 hired labor wage)
p^{k}	Cost of machinery services, excluding interest cost and base wage component of machinery operators' wage
p^{i}	Long-term interest rate
p^{c}	Short-term interest rate
p^{s}	0.1 (seed inputs are stated in 1,000 pesos and the objective function and income rows are stated in 10,000 pesos)

Table 10-5, continued

p_r^f	0.1 (same treatment as seeds)
p_r^a	Regional unit cost of draft animal services
p_d^{wg}	Gravity water
p_d^{wp}	Well water
p_j	Technical progress factor

More variables —

W_g^d Gravity water supply by district

W_p^d Well water supply by district

$\bar{W}_g$ Sector use of gravity water

$\bar{W}_p$ Sector use of pump water

Number of
Constraints

(i) Commodity balances, sectoral and district

33 (a) $y_j X_j + M_j - \alpha_j D^g - E_j + P_j \geq 0$

[Domestic Production] + [Imports] - [Domestic sales] - [Exports]

+ [Adjustment for yield-enhancing technical programs] $\geqslant 0$

176 (b) $y_j^d X_j^d - T_j^d = 0$, each d, j

[District-level production in various techniques] - [Summation activity for charging district price differentials] = 0

(ii) Sectoral and regional labor balances

1 (a) Sectoral wage accounting equation:

$$-SALS + K + \sum_r \lambda_r SALr = 0$$

- [Wage charging activity] + [Accounting activity for employment of machinery operators] + [Regional wage differentials x regional wage accounting activities] = 0

4 (b) Regional wage accounting rows:

$$-SALr + \kappa RESr + \sum_{d \in r} \sum_t dDLt \leq 0,$$

each r

$$- \begin{bmatrix} \text{Regional wage} \\ \text{accounting} \\ \text{activities} \end{bmatrix} + \begin{bmatrix} \text{Reservation wage ratio} \\ \text{x regional farmer em-} \\ \text{ployment activity} \end{bmatrix}$$

$$+ \begin{bmatrix} \text{Sum over districts and} \\ \text{months of regional day} \\ \text{labor employment} \end{bmatrix} \leq 0$$

4 (c) Regional farmer employment accounting rows:

$$-RESr + 3 \sum_{d \epsilon r} \sum_{q} dFLq + \sum_{d \epsilon r} \sum_{t} dFLt = 0, \qquad \text{each } r$$

$$- \begin{bmatrix} \text{Regional farmer} \\ \text{employment} \\ \text{activity} \end{bmatrix} + 3 \begin{bmatrix} \text{Sum over districts} \\ \text{and quarters of} \\ \text{quarterly farmer} \\ \text{employment} \end{bmatrix}^{*} + \begin{bmatrix} \text{Sum over districts} \\ \text{and months of month-} \\ \text{ly farmer employment} \end{bmatrix}^{*} = 0$$

1 (d) Total employment accounting row in man-years:

$$-12LMAN + \sum_{t} LMANt = 0$$

$$-12 \begin{bmatrix} \text{Total employment} \\ \text{in man-years} \end{bmatrix} + \begin{bmatrix} \text{Sum over months of total} \\ \text{employment in man-months} \end{bmatrix} = 0$$

12 (e) Total monthly employment accounting rows in man-months:

$$-2.2LMANt + \sum_{d} dDLt + \sum_{d} dFLq + \sum_{d} dFLt = 0,$$

each t and q such that $t \in q$

---

* In irrigation districts the quarterly contract device is used for farmers, but in non-irrigated districts farmers are assumed to be available on a monthly basis, so that seasonal migration to irrigated areas may occur.

$$-2.2\left[\begin{array}{c}\text{Total employment}\\ \text{in month t}\end{array}\right]^{*}+\left[\begin{array}{c}\text{Sum over districts of day}\\ \text{labor employment in month t}\end{array}\right]$$

$$+\left[\begin{array}{c}\text{Sum over districts of}\\ \text{quarterly farmer employ-}\\ \text{ment in the quarter con-}\\ \text{taining month t}\end{array}\right]+\left[\begin{array}{c}\text{Sum over districts}\\ \text{of monthly farmer}\\ \text{employment}\end{array}\right]=0 .$$

(f) Regional employment balances, by month:

24 (f.1) $\sum_{d\epsilon r} dDLt - MDL3rt \leq L_r,\ r=1,\ 2,\quad \text{each t}$

$$\left[\begin{array}{c}\text{Total employment of}\\ \text{day labor in region}\\ \text{r in month t}\end{array}\right]-\left[\begin{array}{c}\text{Migration of day}\\ \text{labor from Central}\\ \text{Plateau to region}\\ \text{r in month t}\end{array}\right]\leq\left[\begin{array}{c}\text{Pool of landless}\\ \text{labor in region r}\end{array}\right]$$

12 (f.2) $\sum_{d\epsilon r} dDLt + \sum_{r=1}^{2} MDL3rt - MDL43A - MA33t \leq L_3,\ r=3,\ \text{each t}$

$$\left[\begin{array}{c}\text{Total employment}\\ \text{of day labor in}\\ \text{region r=3, month t}\end{array}\right]+\left[\begin{array}{c}\text{Migration out}\\ \text{of region 3}\\ \text{in month t}\end{array}\right]+\left[\begin{array}{c}\text{Migration from}\\ \text{region 4 to}\\ \text{region 3}\end{array}\right]$$

$$+\left[\begin{array}{c}\text{Movement of temporal}\\ \text{farmers into day}\\ \text{labor pool in month t}\end{array}\right]\leq\left[\begin{array}{c}\text{Pool of landless}\\ \text{labor in region 3}\end{array}\right]$$

12 (f.3) $\sum_{d\epsilon r} dDLt + MDL43A - MA44A \leq L_4,\ r=4,\ \text{each t}$

$$\left[\begin{array}{c}\text{Total employment}\\ \text{of day labor in}\\ \text{region r=4,}\\ \text{month t}\end{array}\right]+\left[\begin{array}{c}\text{Migration from}\\ \text{region 4 to}\\ \text{region 3}\end{array}\right]-\left[\begin{array}{c}\text{Transfer of tropical}\\ \text{farmers to day labor}\\ \text{pool}\end{array}\right]$$

$$\leq\left[\begin{array}{c}\text{Pool of landless}\\ \text{labor in region 4}\end{array}\right]$$

* The activities for hiring farmers and day laborers are stated in units of tens of man-days per month (or quarter), and there are 22 working days per month; hence the conversion factor of 2.2 is required in the first term of this equation.

(g) Migration constraints:

12 (g.1) $\sum_{r=1}^{2} MDL3rt - MDL43A - MA33t \leq M_{3t}$ each t

[Bound on monthly migration out of region 3]

1 (g.2) $\sum_{t} \sum_{r=1}^{2} MDL3rt - 12MDL43A - \sum_{t} MA33t \leq M_{3}$

[Bound on annual migration out of region 3]

1 (g.3) $12MDL43A + 12MA44A \leq M_{4}$

[Bound on annual migration out of region 4]

(iii) Sectoral and regional input balances (excl. labor)

1 (a) Short-term credit balance:*

$$\sum_{d} C^{d} - C \leq 0$$

[Sum of district credit counting activities] - [Sectoral interest-charging activity for credit] ≤ 0

1 (b) Machinery services balance:

$$\sum_{d}\sum_{h}\sum_{j} {}^{d}_{hj}X^{d}_{hj} - K \leq 0$$

[Sum of demands for machinery services in cropping activities] - [Activity supplying machinery services] ≤ 0

---

* There are district-level credit balances which sum the demands for credit over cropping activities. There are also bounds on institutional credit allocations by crop which have been made non-operative in the solutions reported here.

1 (c) Balance for charging interest component of machinery services:

$$K - 2.308K' \leq 0$$

$$\left[\begin{matrix}\text{Machinery services in}\\ \text{tens of work-days}\end{matrix}\right] - 2.308 \left[\begin{matrix}\text{Machinery services in}\\ \text{10,000 pesos/year}\end{matrix}\right]^{*} \leq 0$$

1 (d) Sectoral accounting row for use of gravity-fed water:**

$$\sum_d \sum_z \sum_j \gamma_{ij}^{dz} X_{ij}^{dz} - W_g = 0, \quad i = i'$$

$$\left[\begin{matrix}\text{Total demands for}\\ \text{gravity water}\end{matrix}\right] - \left[\begin{matrix}\text{Gravity water}\\ \text{accounting activity}\end{matrix}\right] = 0$$

1 (e) Sectoral accounting row for use of well water:

$$\sum_d \sum_z \sum_j \gamma_{ij}^{dz} X_{ij}^{dz} - W_p = 0, \quad i = i''$$

$$\left[\begin{matrix}\text{Total demands for}\\ \text{well water}\end{matrix}\right] - \left[\begin{matrix}\text{Well water}\\ \text{accounting activity}\end{matrix}\right] = 0$$

1 (f) Sectoral balance for purchased seeds:***

$$\sum_d \sum_j \sigma_j^d X_j^d - S \leq 0$$

$$\left[\begin{matrix}\text{Total demands for}\\ \text{purchased seeds}\end{matrix}\right] - \left[\begin{matrix}\text{Supply of}\\ \text{purchased seeds}\end{matrix}\right] \leq 0$$

---

* The factor 2.308 converts from tens of days to 10,000 pesos per year, given the actual initial cost and lifetime of a typical piece of machinery in Mexico.

** This row and the subsequent one permit experiments with uniform sector-wide changes in the price of water.

*** Both terms in units of thousands of pesos.

4 (g) Regional balances for chemical inputs:*

$$\sum_{d\epsilon r} \sum_{j} \phi_j^d X_j^d - F^r \leq 0, \quad \text{each } r$$

$$\left[\begin{array}{c}\text{Total regional demands}\\ \text{for fertilizers and}\\ \text{pesticides}\end{array}\right] - \left[\begin{array}{c}\text{Regional supply of}\\ \text{fertilizers and}\\ \text{pesticides}\end{array}\right]$$

4 (h) Regional balances for draft animal services:

$$\sum_{d\epsilon r} \sum_{h} \sum_{j} \theta_{hj}^d X_{hj}^d - A^r \leq 0, \quad \text{each } r$$

$$\left[\begin{array}{c}\text{Total regional demands}\\ \text{for services of draft animals}\end{array}\right] - \left[\begin{array}{c}\text{Regional supply of}\\ \text{draft animal services}\end{array}\right]$$

1 (j) Long-term private capital balance:

$$\sum_{d} \lambda_n^d I_n^d - CP \leq 0,$$

$$\left[\begin{array}{c}\text{Costs of investment}\\ \text{activities financed}\\ \text{with private capital}\end{array}\right] - \left[\begin{array}{c}\text{Supply of}\\ \text{private capital}\end{array}\right] \leq 0$$

1 (k) Total long-term capital balance:

$$\sum_{d} \lambda_n^d I_n^d + K' + CP - CT \leq 0, \quad \text{those } n \text{ not in equation (j) above,}$$

$$\left[\begin{array}{c}\text{Costs of investment}\\ \text{activities financed}\\ \text{with public capital}\end{array}\right] + \left[\begin{array}{c}\text{Capital component}\\ \text{of machinery}\\ \text{services}\end{array}\right] + \left[\begin{array}{c}\text{Private}\\ \text{capital}\\ \text{supplied}\end{array}\right]$$

$$- \left[\begin{array}{c}\text{Total capital}\\ \text{supplied}\end{array}\right] \leq 0$$

---

* The indexes d and j on ϕ_j^d indicate that rates of fertilizer use vary over district and crop, but not over other dimensions such as zones or degrees of mechanization. Both terms in this expression are in units of thousands of pesos.

(iv) <u>District-level input balances</u>

204 (a) District Labor balances:

$$\sum_h \sum_j \beta^d_{hjt} X^d_{hj} - \mathrm{dDLt} - \mathrm{dFLt}^* \leqslant 0, \qquad \text{each } d,\ t$$

$$\boxed{\begin{matrix}\text{Demands for labor,}\\ \text{district d, month t}\end{matrix}} - \boxed{\begin{matrix}\text{Day labor hired in}\\ \text{district d, month t}\end{matrix}}$$

$$- \boxed{\begin{matrix}\text{Farmers employed in}\\ \text{district d, month t}\end{matrix}} \leqslant 0$$

17 (b) District credit balances:

$$\sum_h \sum_h \gamma^d_{hj} X^d_{hj} - C^d \leqslant 0, \qquad \text{each } d$$

$$\boxed{\begin{matrix}\text{Demands for credit,}\\ \text{district d}\end{matrix}} - \boxed{\begin{matrix}\text{District credit}\\ \text{counter}\end{matrix}} \leqslant 0$$

10 (c) District gravity water balances:

$$\sum_z \sum_j \delta^{dz}_{ij} X^{dz}_{ij} - W^d \leqslant 0, \qquad \text{each } d, \quad i = i'$$

$$\boxed{\begin{matrix}\text{Demands for gravity}\\ \text{water, district d}\end{matrix}} - \boxed{\begin{matrix}\text{District d activity}\\ \text{for charging costs}\\ \text{of gravity water}\end{matrix}} \leqslant 0$$

4 (d) District well water balances:

$$\sum_z \sum_j \delta^{dz}_{ij} X^{dz}_{ij} - W^d_p \leqslant 0, \qquad \text{each } d, \quad i = i''$$

$$\boxed{\begin{matrix}\text{Demands for well}\\ \text{water, district d}\end{matrix}} - \boxed{\begin{matrix}\text{District d activity for}\\ \text{charging costs of well water}\end{matrix}} \leqslant 0$$

---

* Or dFLq, depending on the district

(v) District resource constraints*

348 (a) Monthly land constraints:

$$X_t^{dz} \leqslant B_t^{dz}, \quad \text{each } d,\ z,\ t$$

$$\left[\begin{array}{c}\text{Land requests in}\\ \text{cropping (units of}\\ \text{X are hectares)}\end{array}\right] \leqslant \left[\begin{array}{c}\text{Land availability by}\\ \text{district, zone, month}\end{array}\right]$$

168 (b) Monthly gravity and pump water constraints:*

$$\sum_z \sum_j \gamma_{ijt}^{dz} X_{ij}^{dz} \leqslant \bar{W}_{it}^{d}, \text{ each } d,\ t,\ i$$

$$\left[\begin{array}{c}\text{Total month t water}\\ \text{demands, district d,}\\ \text{water type i}\end{array}\right] \leqslant \left[\begin{array}{c}\text{Water delivery constraints}\\ \text{district d, water type i}\end{array}\right]$$

14 (c) Annual gravity and pump water constraints:

$$\sum_z \sum_j \gamma_{ij}^{dz} X_{ij}^{dz} \leqslant \bar{W}_{it}^{d},$$

(d) District constraints on farmer and family labor:

44 (d.1) $dFLq \leqslant A_d$, each q, each d with irrigation

36 (d.2) $dFLt - MA33t \leqslant A_d$ each t, each d in region 3 without irrigation

12 (d.3) $dFLt - MA44A \leqslant A_d$ each t, each d in region 4 without irrigation

---

*This statement of the district resource constraints omits the creation of new water supplies and new irrigated land and farmers which occurs when water-enhancing investment occurs. These effects are included in the model.

(vi) <u>Technical progress balances</u>

33 $$\alpha_j^g D^g + E_j - M_j - p_j P_j = 0 , \quad \text{each } j,$$

$$g \text{ such that } j \in \mathcal{g}$$

$$\begin{bmatrix}\text{Total sales on}\\ \text{domestic markets}\end{bmatrix} + \begin{bmatrix}\text{Exports}\end{bmatrix} - \begin{bmatrix}\text{Imports}\end{bmatrix} - \begin{bmatrix}\text{Technical}\\ \text{progress factor}\end{bmatrix} = 0$$

(vii) <u>Income constraints</u>

1 (a) Farmers' profit:

$$\sum_g \sum_s \sum_m \rho_s^g D_{ms}^g + \sum_j p_j^e E_j - \sum_j p_j^m M_j$$

$$- p^{\ell} SALS - p^k \bar{K} p^i K' - p^c CP - p^s S - \sum_r p_r^f F^r$$

$$- \sum_r p_r^a A - \sum_d p_d^{wg} W_g^d - \sum_d p_d^{wp} W_p^d$$

$$- (\Delta p^{wg}) W_g - (\Delta p^{wp} W_p)$$

$$+ \sum_d \sum_j (\Delta p)^d T_j^d \geq Y$$

where $(\Delta p)^{wg}$ and $(\Delta p)^{wp}$ indicate the uniform sector-wide changes in water prices, and $(\Delta p)^d$ indicates the district price differentials by crop.

$$\begin{bmatrix}\text{Gross revenue}\\ \text{from domestic sales}\end{bmatrix} + \begin{bmatrix}\text{Export}\\ \text{earnings}\end{bmatrix} - \begin{bmatrix}\text{Import}\\ \text{Costs}\end{bmatrix}$$

$$- \begin{bmatrix}\text{Total labor}\\ \text{costs}\end{bmatrix} - \begin{bmatrix}\text{Interest on}\\ \text{long-term capital}\end{bmatrix} - \begin{bmatrix}\text{Interest on}\\ \text{Short-term capital}\end{bmatrix}$$

$$- \begin{bmatrix}\text{Seed}\\ \text{costs}\end{bmatrix} - \begin{bmatrix}\text{Chemical}\\ \text{Input costs}\end{bmatrix} - \begin{bmatrix}\text{Draft animal}\\ \text{Services costs}\end{bmatrix}$$

$$- \begin{bmatrix}\text{Gravity water}\\ \text{costs}\end{bmatrix} - \begin{bmatrix}\text{Well water}\\ \text{costs}\end{bmatrix} - \begin{bmatrix}\text{Increments to}\\ \text{gravity water cost}\end{bmatrix}$$

$$- \begin{bmatrix}\text{Increments to}\\ \text{Well water cost}\end{bmatrix} + \begin{bmatrix}\text{District price}\\ \text{differences on crops}\end{bmatrix}$$

1 (b) Farmers' income:

This equation is the same as (vii.a) except that the term

$$+\sum_r a_r \text{RESr} \qquad \text{is added,}$$

where a_r is the regional farmer reservation wage, to serve the purpose of adding farmers' wage income to profits in order to arrive at total farmers' income.

1 (c) Sector Income:

This equation is the same as (vii.a) except that the following term is dropped:

$$-p^{\ell}\ \mathrm{SALS},$$

and the price of machinery services, p^k, is reduced to take out labor costs. These adjustments result in an expression for total sector income which is defined as farmers' income plus wage income of day laborers.

1 (vii) <u>Objective function (maximand)</u>

$$\begin{aligned}
&\sum_g \sum_s \sum_m \omega_s{}^g\, D^g_{ms} + \sum_j p^e_j\, E_j - \sum_j p^m_j\, M_j \\
&- p^{\ell}\mathrm{SALS} - p^k K - p^i CT - p^s\, S - \sum_r p^f_r\, F^r \\
&- \sum_r p^a_r\, A^r - \sum_j p^{wg}_d\, W^d_g - \sum_d p^{wp}_d\, W^d_p \\
&- (\Delta_p{}^{wg})\, W_g - (\Delta_p{}^{wp}) W_p + \sum_d \sum_j (\Delta_p)^d\, T^d_j
\end{aligned}$$

There are differences between this and (vii.a) in the demand function term and in the role of long-term private and public capital. The first term of the objective function is the sum of consumers' and producers' surpluses rather than gross revenue, and total long-term capital is costed, via CT, instead of just private long-term capital.

Sample Tableau

CHAC

Machinery Services

Row Name	Crops: El Bajío 1	2	3	4	Crops: Other Districts 1	2	MAQRd 1	2	3	4	MAQS	MAQKS	SALS	TCS	
SØB											-.1685		-.0195	-r	
SIN1											-.1685	-.12	-.0195		
SIN2											-.1685	-.12	-.0195		
SIN3											-.1560	-.12			
R1MAQ	+						-1								≤ 0
R2MAQ		+						-1							≤ 0
R3MAQ			+						-1						≤ 0
R4MAQ				+						-1					≤ 0
SMAQ					+1	+1	+1	+1	+1	+1	-1				≤ 0
SKMAQ											+1	-2.308			≤ 0
SSAL											+1		-1		≤ 0
STC												+1		-1	≤ 0

The activities MAQS, MAQRd supply 10 days of machine services. MAQS includes charges for 10 days of services for all costs except i) the interest charge on machinery and ii) the base wage for labor. It includes the fixed skill differential of 12.5 pesos per day. The activities MAQKS and TCS charge the interest costs against the income constraints and objective function, respectively. The -.12 coefficients in MAQKS refer to the subsidized interest rate of 12 per cent.

Note that the constant -2.308 implies that MAQKS supplies 10,000 pesos worth of machinery for one year, or the equivalent. For example, a tractor's price is 120,000 pesos, so this activity supplies the tractor for 1/12 of one year (about 20 days). The entry in SØB is zero because the true interest rate is charged in TCS. If the interest rate is 12 percent, the capital charge for 10 days is 520 pesos.

Sample Tableau

CHAC: El Bajío: Water

	CROP GRAV		CROP WELL		RGAGL	RGAGS	RBAGL	RBAGS	GAGS	BAGS		Units
El Bajío Zone	3	4	3	4	3	4	3	4				
SØB					-c	-c	-c′	-c′			(MAX)	10^4
SIN 1-3					-c	-c	-c′	-c′			$\geq Y_1 - Y_3$	10^4
SGAG					+1	+1			-1		= 0 Sector rows to vary water prices	$10^4 xm^3$
SBAG							+1	+1		-1	= 0	
RBGAG	+	+									$\leq B_1$ Annual District bound on gravity water	$10^4 xm^3$
R3GAG	+				-10						≤ 0	$10^4 xm^3$
R4GAG		+				-10					≤ 0 Gravity water charging rows	
RGAGt	+	+									$\leq B_{2t}$ Monthly bounds on gravity water t = 1 - 5, 7, Ø, N, D	$10^4 xm^3$
R3BAG			+					-10			≤ 0	$10^4 xm^3$
R4BAG				+					-10		≤ 0 Pump water charging rows	
RBAGt			+	+							$\leq B_{3t}$ Monthly bounds on pump capacity t = 1 -5, Ø, N, D	$10^4 xm^3$
RBGAG3			+								$\leq B_4$ Annual bounds on water pumped	$10^4 xm^3$
RBGAG4				+							$\leq B_4$ (NB, should be labelled RBBAGz)	

SAMPLE TABLEAU

CHAQUITA

Investment Activities for Increasing

Water Supplies: Central Irrigated Submodel

row \ col.	WIDW	WWØPER	WISD	RHS
SØB		-0.04		
SIN1-3		-0.04		
WSUBr	-400.00		-1,200.00	
WGAGA	-227.21		- 392.70	
WGH2Ør	- 75.72		- 314.16	
WFAMs	-181.55		-2,643.75	
XFAMs	181.55		2,643.75	
SPCAP	25.35		300.00	
WWCURR	181.80	-1.00		
WBWELL	1.00			≤ 2.10
WBDAM			1.00	≤ 0.20

r=1, 2, 3; s=1, 2.

Key to new rows, columns:

Symbols	Identification	Units
WIDW	Activity for Digging Wells	1 well
WWØPER	Activity for Charging Operating Costs on New Wells	10,000 pesos per 10,000 m^3 of water
WISD	Activity for Constructing Small Dams	0.1 dam
WWCURR	Balance for Charging Operating Costs	10,000 m^3 of water
WBWELL	Bound on Well Digging	1,000 wells
WBDAM	Bound on New Small Dams	1,000 dams

11. Risk and Dualism

Risk variables are the major omission on the production side of the model's first version. Perceived risk obviously plays an important role in farmers' decisions, and some attempts were made to incorporate it, but the data were insufficient to support the attempts. New lines of attack have been tentatively formulated, so it is likely that there will be a version of CHAC with risk in due course; in the meantime it is instructive to discuss the reasons why earlier attempts failed. First, it should be noted that district-level changes in cropping patterns implied by the solutions of CHAC generally are not more severe than historical year-to-year changes, particularly in irrigated areas. Quite marked annual changes in planted acreage per crop - often of 50 per cent or more - are observed in these areas. Hence, even without risk variables, in general the model does not appear to violate rough rules of risk-aversion. This may not be so true of more traditional, non-irrigated farms, where the model indicates substantial shifts out of maize and into sorghum when it is assumed that import barriers are weakened.

The first thought on handling risk was to utilize the crop insurance premiums of the national agricultural insurance company. Although proper formulation of risk in an optimizing model leads to a quadratic objective function,* a national crop insurance organization has to make a sensible linearization of the problem, which is valid for marginal changes, in order to derive risk premiums by crop. If it is assumed that such a procedure has been followed, insurance may be specified as a cost of pro-

---

* See the classic work on risk under optimization by Rudolf Freund (1956).

duction, and the observed premiums may be utilized as insurance input coefficients in the cropping activities. Unfortunately, this approach was vitiated by (a) incomplete coverage of the sector in the insurance program, and (b) inconsistency among premiums in areas where there is coverage. If nothing else, this inquiry indicates that it might be fruitful to examine the premium-setting rules in the insurance program.

As a second approach, the method of safety-first constraints was examined. It was concluded that there is little objective basis for establishing appropriate numbers in such constraints, especially in circumstances of highly flexible cropping patterns. It was decided only to impose such constraints after the initial solutions, if they appeared warranted by the results; but, as indicated, it was not necessary to do so.

The third approach, which is not yet fully formulated, may bear fruit eventually. It combines the following elements: (a) the linearization of quadratic forms utilized for the demand specification, (b) risk variables in the objective function, following Freund, and (c) econometric estimation of both objective probabilities in physical yields and prices and subjective preferences toward risk.

Consideration of risk emphasizes the relative inflexibility of non-irrigated agriculture. This is reflected in CHAC in fewer crop choices for non-irrigated agriculture. In the solutions, the difference made by irrigation showed up strongly in terms of responses to price subsidies. When the wheat price was subsidized at 5 percent of the equilibrium price per ton,[*] the long-run elasticity of response[**] was about +5.0. However, when the maize

---

[*] Not to be confused with a guaranteed price level.

[**] This is not the same as the elasticity of supply, for the demand parameters enter into the determination of response.

prize was subsidized a like amount, the elasticity of response was zero. Effectively, farmers with irrigation face many alternatives of nearly equal profitability, and minor pertubations of prices will push them into one of them quickly. On the other hand, farmers without irrigation who grow maize can grow little else easily, and so stronger price changes are required to induce shifts in their cropping patterns.

These results indicate that, at least to a degree, CHAC has captured the dualism of Mexican agriculture. There is another dichotomy on the production side which is not explicit in CHAC, and that is the ejido-private classification of farms. The ejido, one of the products of the Mexican revolution, is the institution of public ownership of farm land. An ejido farmer is granted life-long rights to work his land, but he may not sell it or lease it.* In some locales, the ejido is associated with collective farming. It is generally felt that ejidos are somewhat less efficient than their private counterparts, but available evidence is ambiguous on this point.** The relevant consideration for CHAC is that production costs and yields are defined as averages over geographical areas, and hence these averages include both ejidos and private farms. Since the numbers of farms in each category are stable, their contributions to the averages are stable. CHAC is not addressed to an evaluation of the ejido as an institution, but rather to sector-wide problems of supply, employment, trade, pricing, and resource allocation.

---

* The ejido is the subject of innumerable treatises; perhaps the definitive work on it of recent years is that of Salomón Eckstein (1966).

** See the analyses of agricultural census data for ejidal and non-ejidal tenures by Reed Hertford (1971).

12. Further Research

The process of building CHAC has opened up many lines of further inquiry, some of which have been alluded to in the foregoing sections. One line of work may be summarized under the heading of refining and improving CHAC within the existing framework. Studies are underway or contemplated for a livestock submodel, for a submodel for long-cycle crops, for better estimation of demand functions, for improved estimates of prices in international trade, for estimates of water and fertilizer response curves, for better identification of investment activities, and for incorporation of risk. Some of this new information can be tested, at least initially, with submodels of CHAC. There are also a number of structural refinements which will significantly reduce the size of the model* without any information loss.

Clearly there are separate methodological issues involved in some of these studies. Other basic issues which require more exploration are aggregation, better definition of an income distribution within CHAC, derivation of partial-equilibrium decision rules which are consistent with a general equilibrium framework, and optimal levels of guaranteed prices. For the latter, a small mixed-integer demonstration model has been constructed and is giving reasonable preliminary results. It is planned to extract aggregate supply functions by crop from CHAC, via parametric variations on price variables, and put these into the mixed integer model.

---

* Perhaps by 25 per cent in the number of coefficients.

Partial CHAQUITA Results for 1974 from basic solution 12/6/71

Impact (Percentage Change) on Selected Variables of Policy Changes a/

Variable	Foreign Exchange Premium 15%	Foreign Exchange Premium 30%	Higher Interest Rate 18%	Higher Interest Rate 24%	Higher Wage Rate by 22%	Higher Wage Rate by 44%	Fertilizer Subsidy by 30%	Supply Controls I	Supply Controls II	Supply Controls III	Crop Subsidies Maize	Crop Subsidies Wheat	Crop Subsidies M+W
Maximand	0.9	2.1	-0.7	-1.4	-1.4	-2.8	1.9	-0.1	-0.1	-0.1	1.0	0.2	1.2
Farmers' profits	10.5	24.0	10.2*	17.3*	6.9	-1.7	1.9	10.0	assumed 30.0	50.0	10.4	1.7	12.0
Farmers' income	7.3	16.6	7.8*	20.3	4.7	3.2	1.9	6.3	19.3	32.2	8.3	1.3	9.0
Day labor income	27.5		2.9	-1.0	35.1	84.5	-11.5	0.9	-2.0	-1.0	10.4		
Consumers' surplus	-0.3	-0.7	-2.3	-3.9	-2.1	-2.9	1.9	-1.5	-4.3	-7.3	-0.4		-0.4
Production	2.4	4.6	-1.6	-2.9	-2.4	-3.3	1.9	-0.3	-0.6	-0.9	2.6	1.6	2.2
Net Exports	21.1**	41.0	-3.0	-5.7	-2.8	-5.7	5.0	7.3	42.7	77.1	94.8	52.3	84.4
Employment	4.0	8.7	0.8	1.7	-6.9	-12.0	-3.7	-0.4	-1.0	-1.1	3.8	2.4	3.0
Non-irrig. emp.	3.8	10.6					-2.1	-0.3	-1.7	-1.9	0.7	0.8	
Irrig. emp.	8.6	14.5					-11.0	-0.7	+0.4	+0.1	8.8	4.8	8.2
Long-term capital	1.3	5.5	-6.3	-16.6	6.9	13.1	4.0	-0.3	-1.5	-1.9	6.3	3.4	5.3
Machinery use	1.7	3.8	-5.9	-18.2	8.2	16.0	4.5	-0.3	-1.6	-2.0	6.2	2.4	5.2
Short-term credit	3.6	7.1	-4.1	-5.6	-3.5	-5.3	6.0	-0.1	-0.3	-0.3	3.2	1.7	2.8
Chemical inputs	4.6	11.6					11.0	-0.4	+0.5	+0.3	5.6	5.4	5.4
Subsidy cost+	1050						2000						

Notes to Table:

a/ Expressed in terms of per cent change from 1974 base case

* With higher wage rates, increases in the interest rate cause farmers' profits and income to decrease.

** Corresponds to export growth rate of 10.2 per cent, vs. 6 - 8 per cent in base case

\+ Million 1968 pesos.

12/6/71

Partial CHAQUITA Results

Basic 1968-74 Growth Rates

Solution Values*

	1968		1974 base	Ratio	Growth Rate
Maximand	6995.4	1835	8829	1.262	4.0%
Farmers' profits	887.8	142	1030	1.160	2.5%
Farmers' income	1320.7		1537	1.164	2.6%
Day labor income	78.6		103	1.304	4.5%
* Consumers' surplus	6107.6	1690	7799	1.277	4.2%
Sector income	1399.3	240	1640	1.172	2.7%
Labor force					2.0%
Production					4.65**
Net exports	343.3		509	1.484	6.8%
Employment	2088.0		2460	1.178	2.8%
Machinery use	748.3		887	1.186	2.9%
Short-term credit	1845.2		2426	1.315	4.7%

0.7 pc.

*Base on interest rate of 12%, base wage of 13.5 pesos/day, and no foreign exchange premium.

**Average of Paasche and Laspeyres indexes based on endogenous prices.

* partly back to farmers consumer (40% n 30)

Specify Instruments — descriptive?
normative

Aggregation — agric. by month. — 4 mo. distorts results.

Comp. advantage —

Immed dist. — need detail.

Lessons for Others —

a) other model builders

b) Users — Mexicans

INTERNATIONAL DEVELOPMENT ASSOCIATION | INTERNATIONAL BANK FOR RECONSTRUCTION AND DEVELOPMENT | INTERNATIONAL FINANCE CORPORATION

~~Mr Chenery~~ A1221

DRC file

OFFICE MEMORANDUM

TO: DRC Staff and Optimal Control Project File

DATE: December 3, 1971

FROM: H.K. Kim and C.R. Blitzer H. Kim CRB

DEC 7 1971

SUBJECT: The open Leontief system

1. The purpose of this memo is to set up the open dynamic Leontief input-output model in control theory format. This is intended as a first step in being able to eventually transform more complex and general dynamic linear programming models into corresponding control theoretic formulations. This memo is the first draft of a more complete paper we are preparing on this subject. We hope that you will provide us with numerous comments, criticisms, and suggestions regarding this work as well as the more general applications of control theory to development problems.

2. The general control problem is a composite concept consisting of a dynamic system Σ, a set S_0 of initial states, a target set S_1, a set Γ of admissible controls, a utility functional U, and the statement: "Determine for each initial state (t_0, x_0) an admissible control $u(\cdot)$ which transfers (t_0, x_0) into (t_1, x_1) and which in so doing maximizes our utility functional."

3. A finite-dimensional, discrete-time, linear dynamic system is equivalent to the system of equations

future state control

$$x(t+1) = F(t)\, x(t) + G(t)\, u(t),$$

where

x = n-component state vector,
u = m-component control vector,
F = n $\times$ n matrix,
G = n $\times$ m matrix.

Thus, we shall determine our control theory model by deducing this difference equation from an open dynamic Leontief system.

4. The open dynamic Leontief model can be stated as follows:

$$(1) \quad (I-A)\, x(t) = d(t) + h(t),$$

$$(2) \quad h(t) = B\,(x(t+1) - x(t)),$$

$$t = 0, 1, \ldots, T;$$

Where

x= output levels (n-vector),

d = final demand (n-vector),
h = inputs to investment (n-vector),
A = $n \times n$ Leontief matrix,
B = $n \times n$ capital coefficient matrix.

There are at least three ways in which a linear dynamic system can be deduced from (1) and (2):

(A) Direct Deduction

Substituting (2) into (1) and arranging the terms, we have

(3) $Bx(t+1) = (I-A+B)\,x(t) - d(t)$;

whence, letting $H = I-A+B$, we finally have

(3)' $Bx(t+1) = Hx(t) - d(t)$.

Note that equation (3)' does not define a dynamic system in the sense that since in general B is singular, $x(t+1)$ cannot be uniquely determined from the states $x(t)$ and controls $d(t)$ in time t. This difficulty can be circumvented by rearranging B so that the first m rows are zero and the last $(n-m)$ rows, which represent the capital producing sectors, contain non-zero elements. We can then write the partitioned form of B as

$$B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix},$$

where B_{22} is a square matrix, which is in general non-singular.

Partitioning H in the same way and partitioning $d(t)$ and $x(t)$ accordingly into $d_1(t)$, $d_2(t)$, $x_1(t)$ and $x_2(t)$,

we can rewrite (3)' as (4) and (5).

(4) $H_{11}x_1(t) + H_{12}x_2(t) = d_1(t)$,

(5) $B_{21}x_1(t+1) + B_{22}x_2(t+1) = H_{21}x_1(t) + H_{22}x_2(t) - d_2(t)$.

Assuming H_{11} is nonsingular, we have, from (4),

(6) $x_1(t) = -H_{11}^{-1}H_{12}x_2(t) + H_{11}^{-1}d_1(t)$,

whence, from (5), we derive $x_2(t+1)$ as a linear function of $x_2(t)$, $d(t)$, and $d_1(t+1)$:

(7) $x_2(t+1) = (B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1}(H_{22}-H_{21}H_{11}H_{12})x_2(t)$

$$+ (B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1} H_{21}H_{11}^{-1} d_1(t)$$

$$- (B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1} B_{21}H_{11}^{-1}d_1(t+1)$$

$$- (B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1} d_2(t),$$

where $(B_{22}-B_{21}H_{11}^{-1}H_{12})$ is assumed to be nonsingular.

So far, this formulation of the problem closely resembles the dynamic inverse problem first posed by Leontief (See "The Dynamic Inverse" in Carter, A.P. and A. Brody (eds.), Contributions to Input-Output Analysis, 1969.), in which for computational reasons, the dynamic model is solved through a backward integration scheme starting with values for all $d(t)$ and $x(T+1)$. In our formulation, the same decomposition is done through forward integration, thus avoiding the problem of inconsistent levels of $x(0)$. In other words, given $x(0)$ and all $d(t)$, the model can be solved dynamically without inverting the entire matrix for all time periods. This result is identical to that developed independently by Kendrick in a recent as yet unpublished paper.

Note that equations (7) do not yet define a dynamic system since the state variables $x(t+1)$ are not uniquely defined by $x(t)$ and the set of controls $d(t)$, but required knowledge of controls $d_2(t+1)$. We will now extend the analysis to convert this model into dynamic system required in optimal control theory.

Now, we define

$$(8) \quad d_1(t+1) = u(t).$$

Then, from (7) and (8), we finally have the following linear dynamic system:

$$(9)\quad \begin{pmatrix} d_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1}H_{21}H_{11}^{-1} & (B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1}(H_{22}-H_{21}H_{11}^{-1}H_{12}) \end{pmatrix} \begin{pmatrix} d_1(t) \\ x_2(t) \end{pmatrix}$$

$$+ \begin{pmatrix} I & 0 \\ -(B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1}B_{21}H_{11}^{-1} & -(B_{22}-B_{21}H_{11}^{-1}H_{12})^{-1} \end{pmatrix} \begin{pmatrix} u(t) \\ d_2(t) \end{pmatrix}.$$

Note that in this linear dynamic system d_1 and x_2 are state variables and u and d_2 are control variables. Thus, we have n state variables and n control variables for each period of time.

To sum up, the optimality problem can be posed as one of choosing the feasible path of $u(t)$ and $d_2(t)$ for each time t so as to maximize

$$(10)\quad \sum_{t=0}^{T} (1+\rho)^{-t} f\big(d(t)\big),$$

where ρ is the utility discount rate and $f(\cdot)$ is assumed to be twice differentiable, increasing and strictly concave -- subject to equality contraints (9), initial conditions (11), and terminal conditions (12).

$$(11)\quad \begin{aligned} d_1(0) &= \overline{d_1}(0), \\ x_2(0) &= \overline{x_2}(0), \end{aligned}$$

$$(12)\quad \begin{aligned} d_1(T+1) &= \overline{d_1}(T+1), \\ x_2(T+1) &= \overline{x_2}(T+1). \end{aligned}$$

(B) Capital by Sector of Use

The capital accumulation equations are written (assuming no depriciation)

$$(13)\quad k(t+1) = k(t) + v(t),$$

where

k = n-component vector of sectoral capital stocks,

v = n-component vector of gross investment by sector of use.

The capital constraints are

$$(14)\quad Gx(t) = k(t),$$

where G is a $n \times n$ diagonal matrix of capital-output ratios.

Premultiplying G^{-1} on (14), we derive

$$(15)\quad x(t) = G^{-1}k(t),$$

whence

$$(16)\quad x(t+1) - x(t) = G^{-1}\big(k(t+1) - k(t)\big) = G^{-1}\, v(t).$$

Substituting (16) into (2) and (2) into (1), we derive

$$(17)\quad d(t) = (I-A)x(t) - BG^{-1}v(t).$$

Substituting (17) into (10),

$$(18)\quad U = \sum_{t=0}^{T}(1+\rho)^{-t} f\left\{(I-A)x(t) - BG^{-1}\, v(t)\right\}.$$

The optimality problem is posed as one of choosing the feasible path of gross investment by sector of use, $v(t)$, so as to maximize (18) subject to equality constraints (13) and the initial and terminal conditions (19).

$$(19)\quad k(o) = \overline{k}(o),$$
$$k(T+1) = \overline{k}(T+1).$$

Note that we have n state variables, $k(t)$, and n control variables, $v(t)$, for each time period.

(C) Capital Stock By Sector Of Origin

There is a major computational problem in formulating the optimal control problem in terms of (A) or (B). The number of state variables is the most important determinant of computation time. In order to take advantage of the decomposition properties of control theory, it is necessary to reduce the number of states as much as possible. If the number of producing sectors, n, is large, then brute force linear programming techniques will be less costly for the model defined by (A) and (B). In this section, we will start with our basic open dynamic Leontief model and put it into control theory format while reducing the number of states to the number of capital producing sectors, which in general is much less than present decomposition algorithms.

To begin with, define

$$(20) \quad R = I-A.$$

Partitioning R in the same way as B has been partitioned and partitioning $h(t)$ accordingly into $h_1(t) = 0$ and $h_2(t)$, we derive, from (1),

$$(21) \quad R_{11}x_1(t) + R_{12}x_2(t) = d_1(t),$$

$$(22) \quad R_{21}x_1(t) + R_{22}x_2(t) = d_2(t) + h_2(t).$$

The capital constraints can then be written as

$$(23) \quad \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ k(t) \end{pmatrix},$$

where $k(t)$ is (n-m)-component vector of capital stocks by sector of origin.

From (23), we derive

$$(24) \quad k(t) = B_{21}x_1(t) + B_{22}x_2(t),$$

whence, assuming the existence of B_{22}^{-1},

$$(25) \quad x_2(t) = -B_{22}^{-1}B_{21}x_1(t) + B_{22}^{-1}k(t).$$

Substituting (25) into (22) and solving for $h_2(t)$, we find

$$(26)\quad h_2(t) = (R_{21}-R_{22}B_{22}^{-1}B_{21})\,x_1(t) + R_{22}B_{22}^{-1}k(t) - d_2(t).$$

The capital accumulation equations in this system can be written as

$$(27)\quad k(t+1) = k(t) + h_2(t).$$

Substituting (26) into (27), we find $k(t+1)$ as a function of $k(t)$, $x_1(t)$, and $d_2(t)$:

$$(28)\quad k(t+1) = (I+R_{22}B_{22}^{-1})k(t) + (R_{21}-R_{22}B_{22}^{-1}B_{21})x_1(t) - d_2(t).$$

Finally, if we substitute (25) into (21), then we find

$$(29)\quad d_1(t) = (R_{11}-R_{12}B_{22}^{-1}B_{21})x_1(t) + R_{12}B_{22}^{-1}k(t),$$

whence

$$(30)\quad U = \sum_{t=0}^{T}(1+\rho)^{-t}f\{d_1(t),\ d_2(t)\}$$
$$= \sum_{t=0}^{T}(1+\rho)^{-t}f\{(R_{11}-R_{12}B_{22}^{-1}B_{21})x_1(t) + R_{12}B_{22}^{-1}k(t),\ d_2(t)\}.$$

The optimality problem in this case is then posed as one of choosing the feasible accumulation of capital so as to maximize (30) subject to equality constraints (28) and the initial and terminal conditions:

$$(31)\quad k(o) = \overline{k}(o),$$
$$k(T+1) = \overline{k}(T+1).$$

In this system, we have (n-m) state variables, $k(t)$, and n control variables, $x_1(t)$ and $d_2(t)$ for each time t.

As illustrated by Bruno et al. (1967) and Kendrick (   ), the number of capital goods producing industries is approximately equal to one tenth of the total number of sectors. Thus, if n=100, then the number of state variables can be reduced from 100 to 10 in this system compared with previous models (A) & (B).

Since, as shown by Kendrick ( ), computation time is roughly a linear function of the number of state variables and a less than linear function of the number of control variables, we may be able to solve a relatively large model (100 sectors) in reasonable computation time by using model (C).

6. For practical implementation, it is hard, if not impossible, to specify and estimate our utility function $f(d(t))$ in terms of n commodities. This difficulty can be circumvented by specifying $f(\cdot)$ as a function of total final demand by transforming the model (C) accordingly.

To begin with, we define

$$(32)\quad d_1(t) = \eta(t)q(t),$$

$$(33)\quad d_2(t) = \xi(t)q(t),$$

where $q(t) = \sum_i d_i(t)$ = a scalar representing aggregate final demand,

$\eta(t)$ = m-component vector,

$\xi(t)$ = (n-m)-component vector.

At this stage we assume that both $\eta(t)$ and $\xi(t)$ are exogenously given. These fixed proportions can be parametrically altered for each time t to test alternative demand patterns. This could provide an important link with an econometric policy model of the economy which has an endogenous demand system.

Substituting (33) into (28), we derive the following capital accumulation equation:

$$(34)\quad k(t+1) = (I + R_{22}B_{22}^{-1})k(t) + (R_{21}-R_{22}B_{22}^{-1}B_{21})x_1(t) - \xi(t)q(t).$$

Our optimality problem is then stated as follows:

Choose the feasible accumulation of capital so as to maximize

$$(35)\quad \sum_{t=0}^{T}(1+\rho)^{-t}f(q(t)),$$

subject to equality constraints (34) and the initial and terminal conditions (31).

In this formulation, we have (n-m) state variables, k(t), and (m+1) control variables, $x_1(t)$ and q(t) in each time period t. In comparison with model (C), we have reduced the number of control variables from n to (m+1). Note also that the specification of f(q(t)) is not difficult or uncommon. For example, we may simply use a constant elasticity of marginal utility function, i.e., $f(q) = \frac{1}{1-\sigma} q^{1-\sigma}$. Stoleru (1970) has demonstrated a way of estimating the elasticity of marginal utility, σ, and the utility discount rate, ρ, from empirical data.

We now come back to $\eta(t)$ and $\xi(t)$, which are assumed to be given. We may be able to link our control theory model, which is a production oriented model, to a macroeconometric model, which is distribution and policy oriented, by inserting these values of $\eta(t)$, $\xi(t)$ for each time t into the control model which are derived from the demand system of the policy model.

7. In this paper, we have attempted to put only the open dynamic Leontief model for a closed economy into control theoretic format. In the next stages of this research, we shall attempt to expand the analysis to include dynamic Leontief-type models with endogenous foreign trade, labor inputs and human capital formation, and inequality constraints. When this task is completed, we will be able to solve a wide variety of multisector planning models using control theory algorithms and test the advantages of our approach over standard linear programming methods.

DRC Staff and Optimal Control Project File December 3, 1971

REFERENCES

Bruno, Michael; Mordecai, Fraenkel, and Christopher Dougherty (1968). "Dynamic Input-Output, Trade and Development," (memo) (Jerusalem, Israel: Bank of Israel and Hebrew University, July). ~~Forthcoming in~~ A. Carter and A. Brody, Applications of Input Output Analysis, 1969.

Kendrick, David. Mathematical Methods in Economic Planning, to be published by Holden-Day.

Leontief, Wassily W. (1968). "The Dynamic Inverse" in Carter, A.P. and A. Brody (eds.), Contributions to Input-Output Analysis, 1969.

Stoleru, Lionel (1970). L'Équilibre et La Crouissance Économiques, Dunod, Paris.

DRC
Tui
3-30

The attached tables are the summary of the computational results for "A Model for Evaluating Investment Projects in the Mechanical Engineering Sector", to be discussed on Friday, December 10, 1971.

Computational Results

In order to seek an alternative way of calculating the solution to the straightforward mixed integer programming problem and for purposes of comparison, we have computed seven different costs for 120 products using each set of parameter values, assuming that each item is produced in specialized shops producing only the single item.

Seven different concepts of cost which have been utilized are:

Imports

$$(1)\quad (\text{MC IMP})_1 = C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} W_n$$

$$(2)\quad (\text{MC LOW})_1 = C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} P_n,$$

$$\text{where } P_n = \text{Min}\,(MC_n, W_n)$$

$$(3)\quad (\text{A/A WD})_1 = \left[\sum_{k=1}^{R^*} \left\{F_k\, U\, (H_{k1} B_1/G_k)\right\}\right] \div B_1 + C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} AC_n,$$

$$\text{where } AC_n = \text{Min}\left[(\text{A/A WD})_n, W_n\right]$$

$$(4)\quad (\text{AC 2 TH})_1 = \sum_{k=1}^{R^*} (F_k/2.0)\, H_{k1} + C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} AC_n,$$

$$\text{where } AC_n = \text{Min}\left[(\text{AC 2 TH})_n, W_n\right]$$

Unbounded

$$(5)\quad (\text{A/A UB})_1 = \left[\sum_{k=1}^{R^*} \left\{F_k\, U\, (H_{k1} Y_1/G_k)\right\}\right] \div Y_1 + C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} AC_n,$$

$$\text{where } AC_n = \text{Min}\left[(\text{A/A UB})_n, W_n\right]$$

$$Y_1 = \text{smallest common multiple of } G_k/H_{k1}$$

$$(6)\quad (\text{A/M WD})_1 = \left[\sum_{k=1}^{R^*} \left\{F_k\, U\, (H_{k1} \cancel{Y_1}/G_k)\right\}\right] \div B_1 + (\text{MC LOW})_1$$

Bl

$$(7)\quad (A/M\ UB)_1 = \left[\sum_{k=1}^{R^*} \left\{ F_k\ U\ (H_{k1}\ Y_1/G_k) \right\}\right] \div Y_1 + (MC\ LOW)_1.$$

While (1) is marginal cost at import prices, (2) is true marginal cost which is not higher than (1). While (3) is average cost at the level of B_1, (5) is average cost at the level of the least common multiple of G_k/H_{k1}. From (5) note that the expression $\left[\sum_{k=1}^{R^*} \left\{ F_k\ U\ (H_{k1}\ Y_1/G_k) \right\}\right] \div Y_1$ is equivalent to:

$$\sum_{k=1}^{R^*} \left\{ (F_k/G_k).\ H_{k1} \right\} .$$

In other words, (5) is the average cost when all shops required for the production of 1'th item are operating at full capacity. (4) is average cost at 2,000 effective shop hours utilization. (6) is same as (3) and (7) is same as (5), except the pricing of intermediate inputs.

The ratios of domestic production costs to import costs are computed and the comparative advantage rankings are analyzed, utilizing the various definition of costs, for 9 different sets of parameter values (case B, case C, case D, case E, case F, case G, case H, case HD and case DD).

The computational results based on "MC LOW", "A/A WD", and "A/A UB" of case D are summarized on Table 1.

As shown on Table 1, we can easily determine the optimum level of production for most of the items, based on the three cost data, i.e., $\hat{x} = B_1$ if both $(MC\ LOW)_1$ and $(A/A\ WD)_1 < W_1$ and $\hat{x} = 0$ if $(MC\ LOW)_1 > W_1$.

Additional computations are required to determine the optimum level of production, if $(MC\ LOW)_1 < W_1$ and $(A/A\ WD)_1 > W_1$. In case D, there are only 3 items which fall into such category. They are "Ball Mill", "Coal Cutter" and "Portable Air Compressor". Appendix I summarizes the additional calculations to determine the optimum level of production for these products.

In order to summarize the responses of the ratios of domestic production costs to import costs and the comparative advantage rankings to changes in parameter values, simple correlation coefficients and rank correlation coefficients between various combinations of case B, case C, case D, case E, case F, case G, case H, case HD and case DD have been computed, utilizing each cost concept. Correlations based on "MC LOW", "A/A WD", and "A/A UB" are shown on ~~the footnote of~~ Table 2.

Table 5 shows the comparison between the cost advantage ranking of 34 product classifications made by KIST study [9] and the comparative advantage ranking of 120 product classifications based on "A/A WD" of case D.

Table 6 is a schematic presentation of the technology matrix of the Korean mechanical engineering sector. Resource element and shop hour requirement matrix includes only shared shops.

Table 7 shows comparative advantage ranking based on "A/A WD" of case D by industry.

Table 1

Optimum Level of Production, Comparative Advantage Ranking, and TC/OC in Korea's Mechanical Engineering Sector: Case D[/1]

Product	Optimum Level of Production (M/T)	MC LOW[/2]		A/A WD[/3]		A/A UB[/4]		Remarks[/6]	
		Rank	TC/OC[/5]	Rank	TC/OC	Rank	TC/OC	Current Status	TFYP[/7]
Letter Press	150	1	0.08	1	0.11	1	0.09		
Other fasteners	100	2	0.09	2	0.11	2	0.10		
Machine screws	1,000	3	0.12	3	0.13	3	0.12	Dom. Prod.	
Hand Press	50	4	0.13	9	0.27	4	0.13		
Bolts and nuts	2,000	5	0.14	4	0.14	5	0.14		Export
Rivets	500	6	0.15	5	0.15	6	0.15	Dom. Prod.	

/1 Parameter values for Case D are:

CRF (Capital Recovery Factor) = 0.2351 (equivalent to discount rate 20% and life 10 years)

Exchange Rate = 450 Won/$1

Wage Rate = 150 Won/1 hour

B_1 = KIST Estimate

/2 $$(\text{MC LOW})_1 = C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} P_n,$$

where $P_n = \min(MC_n, W_n)$

/3 $$(\text{A/A WD})_1 = \left[\sum_{k=1}^{R^*} \{F_k U (H_{k1} B_1/G_k)\}\right] \div B_1 + C_1 + \sum_{k=1}^{R^*} V_k H_{k1} + \sum_{n=1}^{120} A_{n1} AC_n,$$

where $AC_n = \min\left[(\text{A/A WD})_n, W_n\right]$

/4 $$(\text{A/A UB})_1 = \left[\sum_{k=1}^{R^*} \{F_k U (H_{k1} y_1/G_k)\}\right] \div y_1 + \sum_{n=1}^{20} A_{n1} AC_n,$$

where y_1 = smallest common multiple of G_k/H_{k1}

$AC_n = \min\left[(\text{A/A UB})_n, W_n\right]$,

/5 OC = import cost

/6 "Domestic production" implies that the item is not imported at all and "Import" means that the item is not produced domestically at all.

/7 The target year of Third Five Year Plan is 1976.

Table 1 (cont'd)

Product	Optimum Level of Production	MC Low		Bounded A/A WD		Unbounded A/A UB		Remarks	
	(M/T)	Rank	TC/OC	Rank	TC/OC	Rank	TC/OC	Current Status	TFYP
Steam Boiler 1 (up to 10 kg/cm^2)	3,000	7	0.20	6	0.20	7	0.20		
Knitting Machine	5,500	8	0.22	7	0.23	8	0.23	Export	Export
Steam Boiler 2 (over10kg/cm^2)	3,000	9	0.24	8	0.25	9	0.25		
Centrifugal Pump 3 (over 200mm dia.)	750	10	0.26	10	0.29	10	0.27	Export	
Offset Printing Machine	180	11	0.27	16	0.41	11	0.27		
Centrifugal Pump 2 (50-200mm dia.)	220	12	0.31	13	0.35	12	0.32		
Oil Stove 1 (without burner)	550	13	0.31	12	0.33	14	0.32		
Household Oven	400	14	0.31	11	0.32	13	0.32		
Fire Extinguisher	155	15	0.32	14	0.35	16	0.35		
Stationary Air Compressor	1,260	16	0.33	15	0.38	15	0.33		
Conveyor	200	17	0.34	31	0.60	17	0.35		
Jib Crane	84	18	0.37	51	0.90	18	0.39		
Blower	355	19	0.38	20	0.47	19	0.39		
Concrete mixer	260	20	0.39	37	0.69	20	0.40		
Dyeing machine	600	21	0.40	17	0.43	21	0.40		
Crusher	400	22	0.40	39	0.72	22	0.40		
Ferrous fittings	5,000	23	0.40	23	0.41	18	0.43		
Rice and barley polishing machine	56	24	0.42	24	0.43	34	0.63		
Crank press	650	25	0.43	25	0.43	30	0.57		
Roll crusher	291	26	0.43	43	0.76	27	0.45		
Hydraulic press	750	27	0.43	24	0.51	26	0.44		

Table 1 (cont't)

Product	Optimum Level of Production (M/T)	MC Low		A/A WD		A/A UB		Remarks	
		Rank	TC/OC	Rank	TC/OC	Rank	TC/OC	Current Status	TFYP
Hydraulic pump	500	28	0.46	23	0.50	30	0.48		
Switchboard and control panel	240	29	0.46	19	0.47	28	0.47		
Air hammer	1,050	30	0.47	29	0.56	29	0.47		
Chain and sprocket	3,000	31	0.47	22	0.50	31	0.49		
Household boiler	4,000	32	0.49	21	0.49	32	0.49		
Loom	6,000	33	0.50	25	0.52	33	0.50	Export	Export
M. T. #2 /1	4,234	34	0.51	26	0.53	35	0.52	Export	Export
Spinning machine	900	35	0.51	27	0.55	34	0.52		
Coil springs	2,000	36	0.55	28	0.55	36	0.55		
Ship 2 (Steel ship 500-20,000 G/T)	250,000 /2	37	0.58	33	0.62	39	0.62	Export	Export
Bronze valve	500	38	0.59	32	0.62	38	0.61		
Tank, vessel, tower	2,003	39	0.59	35	0.63	37	0.60		
Sprayers /3	59	40	0.61	42	0.76	40	0.63	Dom.Prod.	
Truck for special purpose	2,763	41	0.61	41	0.74	41	0.64		
Pole transformer	1,850	42	0.67	36	0.67	42	0.67	Export	Export
Ship 3 (Steel ship 20,000-100,000 G/T)	350.000 /4	43	0.68	40	0.73	46	0.73	Export	Export
Power sprayer	38	44	0.68	52	0.90	43	0.70	Dom.Prod.	
Overhead traversing crane	220	45	0.69	54	0.94	49	0.76		

/1 Machine tool #2 produced by the joint production activity: Engine lathe (high quality), Shaper, Planer, Boring machine, Radial drilling machine, and Milling machine.

/2 G/T

/3 Sprayer produced by the joint production activity: Power sprayer and Manual sprayer

/4 G/T

Table 1 (cont'd)

Product	Optimum Level of Production (M/T)	MC Low		A/A WD		A/A UB		Remarks	
		Rank	TC/OC	Rank	TC/OC	Rank	TC/OC	Current Status	TFYP
Direct current electric motor	375	46	0.70	46	0.81	44	0.71		
Electric motor 2 (1-50 Hp)	2,300	47	0.71	38	0.71	45	0.71		
Bicycle	25,000	48	0.74	44	0.76	47	0.76		
Filter	100	49	0.74	57	0.98	50	0.76		
Rotary press machine	62	50	0.75	55	0.96	48	0.76		
Power transformer	950	51	0.77	45	0.79	51	0.77	Export	Export
Centrifugal pump 1 (up to 50 mm dia.)	230	52	0.77	48	0.84	52	0.78		
Hydraulic components	500	53	0.77	49	0.84	54	0.81		Export
M.T. #1 /1	516	54	0.78	53	0.91	53	0.79	Export	Export
Diesel engine 3 (for truck and bus)	4,650	55	0.80	50	0.84	56	0.83		
Ball Mill	810 /2	56	0.81	60	1.02	55	0.83		
Generator	6,600	57	0.83	47	0.84	57	0.83	Import	Export
Coal cutter	0 /2	58	0.89	108	3.40	58	0.91		
Electric motor 1 (Fractional HP)	65	59	0.92	58	0.99	59	0.93		
Portable air compressor	170 /2	60	0.93	62	1.05	60	0.93		Export
Diesel engine 2 (for marine and industrial use)	17,190	61	0.93	56	0.97	62	0.96		
Air conditioner	500	62	0.93	59	1.00	61	0.96		Export
Electric motor 3 (50-500 Hp)	0	63	1.01	61	1.02	59	0.93		
Diesel engine 1 (for marine and industrial use, 20-200 HP)	0	64	1.03	63	1.07	65	1.07	Import	

/1 Machine tool #1 produced by the joint production activity: Engine lathe (low quality), Shaper, Vertical drilling machine, and Milling machine

/2 For the calculations of the optimum solution, see Appendix I

Table 1 (cont'd)

Product	Optimum Level of Production (M/T)	MC Low		A/A WD		A/A UB		Remarks	
		Rank	TC/OC	Rank	TC/OC	Rank	TC/OC	Current Status	TFYP
Brass fittings	0	65	1.03	64	1.10	64	1.06		
Huller	0	66	1.06	84	1.61	68	1.09		
Arc welder	0	67	1.07	65	1.10	66	1.08		
Circuit breaker 2 (for high tension, over 600V)	0	68	1.08	66	1.11	67	1.08	Import	
Truck	0	69	1.12	67	1.18	69	1,17	Dom.Prod.	
Sewing machine 1 (Household use)	0	70	1.15	72	1.31	77	1.31	Export	Export
Motorcycle	0	71	1.18	68	1.23	70	1.23		
Differential gear and rear axle	0	72	1.20	77	1.45	74	1.27		
Passenger car	0	73	1.20	69	1.24	71	1.23	Dom.Prod.	
Coal briquetting machine	0	74	1.22	89	1.66	72	1.25		
Radiator	0	75	1.26	70	1.28	73	1.26		
Heavy gear	0	76	1.27	74	1.36	75	1.30		
Small engine (up to 15 HP)	0	77	1.28	76	1.42	78	1.31	Import	
Industrial heat exchanger	0	78	1.29	73	1.35	79	1.31		
Distribution transformer	0	79	1.30	71	1.31	76	1.30	Export	Export
Gasoline engine 1 (for passenger car)	0	80	1.32	75	1.40	82	1.36	Import	
Sewing machine 2 (industrial use)	0	81	1.32	81	1.56	83	1.46	Export	Export
Header and upsetting machine	0	82	1.34	102	2.46	81	1.36		
Industrial fan	0	83	1.34	79	1.53	80	1.35		
Piston, piston ring, and piston pin	0	84	1.41	86	1.64	84	1.48		
Gasoline engine 2 (for truck and bus)	0	85	1.44	90	1.72	86	1.49	Import	

Table 1 (cont'd)

Product	Optimum Level of Production (M/T)	MC Low Rank	MC Low TC/OC	A/A WD Rank	A/A WD TC/OC	A/A UB Rank	A/A UB TC/OC	Remarks Current Status	Remarks TFYP
Plastic moulding machine	0	86	1,44	94	1.97	87	1.49		
Typewriter	0	87	1.47	78	1.51	85	1.48		
Refrigerator	0	88	1.50	83	1.57	91	1.55	Export	
Carburetor	0	89	1.51	85	1.61	88	1.52		
Leaf springs	0	90	1.52	80	1.53	89	1.53	Dom. Prod.	
Passenger train car	0	91	1.52	82	1.56	71	1.23	Dom. Prod.	
Bus	0	92	1.53	87	1.65	92	1.61	Dom. Prod.	
Freight car	0	93	1.61	88	1.66	93	1.63	Dom. Prod.	
Transmission	0	94	1.66	91	1.82	94	1.71		
Oil Stove 2 (with burner)	0	95	1.76	92	1.85	95	1.83		
Three wheel truck	0	96	1.77	93	1.88	96	1.84	Dom. Prod.	
Power thresher	0	97	1.85	99	2.15	97	1.90		
Brake	0	98	1.93	95	2.11	99	1.99		
Washing machine	0	99	1.96	98	2.13	98	1.99		Export
Construction equiptments/1	0	100	1.98	96	2.11	100	2.05	Import	
Hand tools	0	101	1.99	100	2.33	102	2.33		Export
Electric fan	0	102	2.05	97	2.12	101	2.11	Dom. Prod.	
Power tiller	0	103	2.38	101	2.45	103	2.43	Export	Export
Ship 1 (Steel ship up to 500 G/T)	0	104	2.45	104	2.64	105	2.62	Export	Export
Ball bearings	0	105	2.56	103	2.63	104	2.59	Export	Export
Flour mill	0	106	2.76	112	4.22	106	2.85		
Light gear	0	107	2.99	106	3.21	107	3.04		

/1 Construction equipment produced by the joint production activity: Bulldozer, Grader, Scraper, Power shovel, Front loader, and Road roller.

Table 1 (cont'd)

Product	Optimum Level of Production (M/T)	MC Low		A/A WD		A/A UB		Remarks	
		Rank	TC/OC	Rank	TC/OC	Rank	TC/OC	Current Status	TFYP
Tractor	0	108	3.01	105	3.20	108	3.09		
Cast iron and steel valve	0	109	3.04	107	3.37	109	3.14		
Clutch	0	110	3.22	110	3.88	111	3.31		
Roller bearings	0	111	3.25	109	3.44	110	3.30	Export	Export
Electrical wire-rope hoist	0	112	3.64	111	3.99	112	3.64		
Mine car	0	113	4.75	119	14.95	113	4.91		
Electrical chain hoist	0	114	5.58	115	8.95	114	5.60		
Injection nozzle	0	115	5.79	113	5.98	115	5.93		
Cement blocking machine	0	116	6.50	116	11.38	116	6.69		
Elevator	0	117	6.77	114	7.04	117	6.90		
Blender	0	118	12.17	117	12.87	118	12.40		
Circuit breaker 1 (for low tension, up to 600V)	0	119	14.54	118	14.75	119	14.54		
Wood ship	0	120	18.69	120	18.95	120	18.74		

SIMPLE CORRELATION COEFFICIENTS

	MC Low	A/A WD	A/A UB
MC Low			
A/A WD	0.936		
A/A UB	1.000	0.938	

RANK CORRELATION COEFFICIENTS

	MC Low	A/A WD	A/A UB
MC Low			
A/A WD	0.972		
A/A UB	0.999	0.973	

Table 2

Comparison of TC/OC and Comparative Advantage Ranking Based on "MC Low": Case B, Case C, Case D, Case E, Case F, Case G, Case H, Case HD, and Case DD/1

(A) SIMPLE CORRELATION COEFFICIENTS

	Case B	Case C	Case D	Case E	Case F	Case G	Case H	Case HD
Case C	0.997							
Case D	0.999	0.999						
Case E	0.993	0.985	0.989					
Case F	0.992	0.997	0.996	0.971				
Case G	0.998	0.992	0.995	0.997	0.983			
Case H	0.989	0.979	0.984	0.997	0.966	0.997		
Case HD	0.999	0.999	1.000	0.989	0.996	0.995	0.984	
Case DD	0.999	0.999	1.000	0.989	0.996	0.995	0.984	1.000

/1 Parameter values for the various cases are as follows.

	CRF	Exchange rate	Wage rate	B_1
Case B	0.3206/2	450 won/$1	150 won/1 hour	KIST Estimate
Case C	0.1598/3	450 won/$1	150 won/1 hour	KIST Estimate
Case D	0.2351/4	450 won/$1	150 won/1 hour	KIST Estimate
Case E	0.2351	300 won/$1	150 won/1 hour	KIST Estimate
Case F	0.2351	600 won/$1	150 won/1 hour	KIST Estimate
Case G	0.2351	450 won/$1	225 won/1 hour	KIST Estimate
Case H	0.2351	450 won/$1	300 won/1 hour	KIST Estimate
Case HD	0.2351	450 won/$1	150 won/1 hour	KIST Estimate ÷ 2
Case DD	0.2351	450 won/$1	150 won/1 hour	KIST Estimate x 2

/2 Capital Recovery Factor 0.3206 is equivalent to discount rate 30% and life 10 years.

/3 Capital Recovery Factor 0.1598 is equivalent to discount rate 10% and life 10 years.

/4 Capital Recovery Factor 0.2351 is equivalent to discount rate 20% and life 10 years.

Table 2 (cont'd)

(B) RANK CORRELATION COEFFICIENTS

	Case B	Case C	Case D	Case E	Case F	Case G	Case H	Case HD
Case C	0.995							
Case D	0.998	0.999						
Case E	0.995	0.994	0.996					
Case F	0.996	0.997	0.997	0.989				
Case G	0.997	0.997	0.998	0.997	0.994			
Case H	0.993	0.993	0.994	0.995	0.988	0.998		
Case HD	0.998	0.999	1.000	0.996	0.997	0.998	0.994	
Case DD	0.998	0.999	1.000	0.996	0.997	0.998	0.994	1.000

Table 3

Comparison of TC/OC and Comparative Advantage Ranking Based on "A/A WD": Case B, Case C, Case D, Case E, Case F, Case G, Case H, Case HD, and Case DD

(A) SIMPLE CORRELATION COEFFICIENTS

	Case B	Case C	Case D	Case E	Case F	Case G	Case H	Case HD
Case C	0.991							
Case D	0.998	0.998						
Case E	0.996	0.981	0.991					
Case F	0.988	0.998	0.996	0.973				
Case G	0.998	0.991	0.997	0.997	0.986			
Case H	0.993	0.981	0.989	0.997	0.972	0.998		
Case HD	0.982	0.955	0.971	0.980	0.954	0.973	0.969	
Case DD	0.975	0.994	0.986	0.964	0.990	0.980	0.969	0.918

(B) RANK CORRELATION COEFFICIENTS

	Case B	Case C	Case D	Case E	Case F	Case G	Case H	Case HD
Case C	0.995							
Case D	0.998	0.998						
Case E	0.996	0.994	0.996					
Case F	0.997	0.997	0.998	0.990				
Case G	0.997	0.996	0.998	0.997	0.995			
Case H	0.994	0.993	0.994	0.995	0.990	0.999		
Case HD	0.993	0.985	0.990	0.989	0.988	0.989	0.987	
Case DD	0.992	0.998	0.996	0.991	0.995	0.994	0.990	0.977

Table 4

Comparison of TC/OC and Comparative Advantage Ranking Based on "A/A UB":
Case B, Case C, Case D, Case E, Case F, Case G, Case H, Case HD, and Case DD

(A) SIMPLE CORRELATION COEFFICIENTS

	Case B	Case C	Case D	Case E	Case F	Case G	Case H	Case HD
Case C	0.997							
Case D	0.999	0.999						
Case E	0.993	0.984	0.989					
Case F	0.992	0.997	0.995	0.971				
Case G	0.998	0.992	0.995	0.997	0.983			
Case H	0.990	0.978	0.985	0.997	0.965	0.997		
Case HD	0.999	0.999	1.000	0.989	0.995	0.995	0.985	
Case DD	0.999	0.999	1.000	0.989	0.995	0.995	0.985	1.000

(B) RANK CORRELATION COEFFICIENTS

	Case B	Case C	Case D	Case E	Case F	Case G	Case H	Case HD
Case C	0.995							
Case D	0.998	0.999						
Case E	0.995	0.995	0.996					
Case F	0.996	0.996	0.997	0.989				
Case G	0.997	0.997	0.998	0.997	0.994			
Case H	0.994	0.993	0.994	0.995	0.988	0.998		
Case HD	0.998	0.999	1.000	0.996	0.997	0.998	0.994	
Case DD	0.998	0.999	1.000	0.996	0.997	0.998	0.994	1.000

Table 5

Comparison of Cost Advantage Ranking:
KIST Study vs. "A/A WD" of Case D

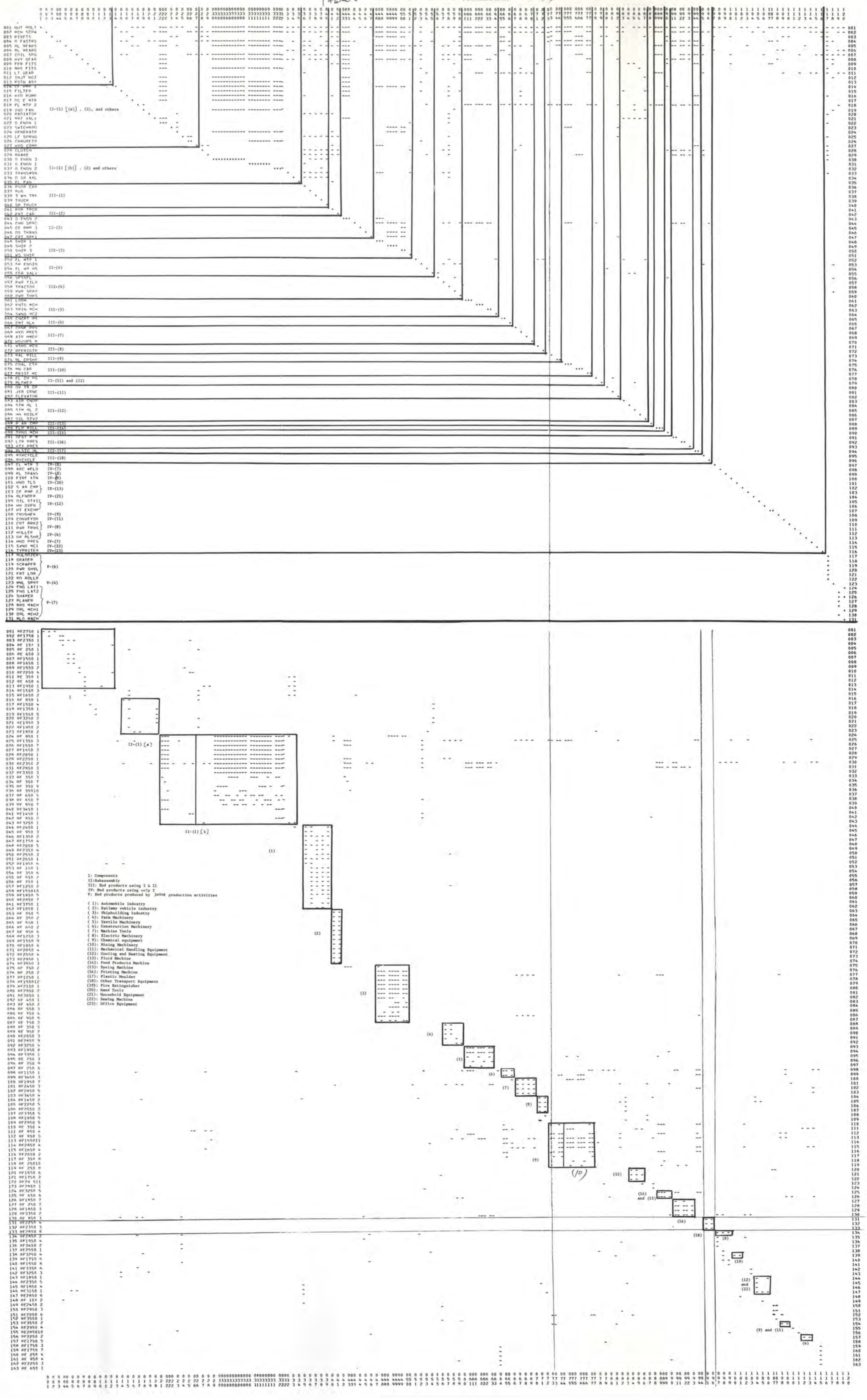
KIST Study Ranked by Cost Advantage		KIST Overall Ranking/2	Ranking Based on "A/A WD" of Case D/3
Ranking/1	Product		
1	Ships	1	30, 40, 104, 106, 120
2	Foundry products	2	74, 77, 86
3	Small work/passenger vehicles	3	44, 68
4	Cutlery, Hand tools	5	100
5	Textile machinery	6	8, 17, 27
6	Machine tools	7	9, 24, 25, 26, 29, 64, 65, 102
7	Sewing machines	12	72, 81
8	Trucks and buses	4	41, 67, 87, 93
9	Electric motors	11	38, 46, 58, 61
10	Boilers	15	6, 8, 21,
11	Toys and Sporting goods	18	
12	Mineral crushing machinery	28	39, 43
13	Printing machinery	30	1, 16, 55
14	Paper machinery	31	
15	Mechanical measuring eq.	32	
16	Blowers and fans	10	20, 79, 97
17	Valves and fittings	14	23, 32, 64, 107
18	Watches and clocks	16	
19	Bearings	17	103, 109,
20	Power transmissions	22	91
21	Springs	24	28, 80,
22	Food machinery	33	112,
23	Pumps and compressors	20	10, 13, 15, 23, 48, 62
24	Heating and cooling equipment	23	12, 59, 73, 92
25	Crane and Hoist	27	51, 54, 111, 115
26	Wood working machinery	29	
27	Construction equipment	8	31, 37, 96, 105, 116
28	Farm machinery	9	24, 35, 42, 52, 84, 99, 101
29	Internal combustion engines	19	50, 56, 63, 75, 76, 90,
30	Office machines	21	78
31	Fasteners	26	2, 3, 4, 5
32	Railway vehicles	25	82, 88,
33	Home appliances	34	11, 83, 98, 117
34	Automobiles	13	69

/1 Ranking of total 34 product classifications.

/2 Based on 11 criteria including cost advantage.

/3 Ranking of total 120 product classification.

Table 6 TECHNOLOGY MATRIX OF KOREAN MECHANICAL ENGINEERING SECTOR



I: Components
II: Subassembly
III: End products using I & II
IV: End products using only I
V: End products produced by joint production activities

(1): Automobile Industry
(2): Railway vehicle industry
(3): Shipbuilding industry
(4): Farm Machinery
(5): Textile Machinery
(6): Construction Machinery
(7): Machine Tools
(8): Electric Machinery
(9): Chemical equipment
(10): Mining Machinery
(11): Mechanical Handling Equipment
(12): Cooling and Heating Equipment
(13): Fluid Machine
(14): Food Products Machine
(15): Dyeing Machine
(16): Printing Machine
(17): Plastic Moulder
(18): Other Transport Equipment
(19): Fire Extinguisher
(20): Hand Tools
(21): Household Equipment
(22): Sewing Machine
(23): Office Equipment

Table 7

Comparative Advantage Ranking ("A/A WD" of Case D) By Industry

Industry	Ranking (total 120 products)	
	TC/OC < 1	TC/OC > 1
I. Common Components	2, 3, 4, 5, 23, 28,	64, 74, 86, 103, 106, 109, 113
II.-(1)(a). Subassembly for Automobile Industry (a) and others	19, 23, 32, 38, 46, 47, 48, 49, 57,	63, 70, 79, 80, 85
II-(1)(b). Subassembly for Automobile Industry (b) and others	50	75, 77, 90, 91, 95, 97, 110
III-(1) Automobile Industry	41,	67, 69, 87, 93
III-(2) Railway Vehicle Ind.		82, 88
II-(3) Subassembly for Shipbuilding Industry	10, 22, 56,	71, 118
III-(3) Shipbuilding Ind.	33, 40,	104, 120
II-(4) Subassembly for Farm Machinery	58	76, 107, 111
Farm Machinery		
III-(4)	35, 52,	99, 101, 105
IV-(4)	24,	84,
V-(4)	42	
III-(5) Textile Industry	7, 25, 27	81
Construction Machinery		
III-(6)	37,	116
V-(6)		96
Machine Tools		
III-(7)	24, 25, 29	102
IV-(7)	9,	65
V-(7)	26, 53	
Electrical Machinery		
III-(8)		83, 98
IV-(8)	36, 45,	61, 66
Chemical Equipment		
III-(9)	43	60
IV-(9)	39	

Table 7 (cont'd)

Industry	Ranking (total 120 products)	
	TC/OC < 1	TC/OC > 1
III-(10) Mining Machinery		89, 108, 119
II-(11) Subassembly for Mechanical Handling Equip.		115
Mechanical Handling Equipment		
III-(11)	51, 54	114
IV-(11)	31	
II-(12) Subassembly for Cooling and Heating Equip.	20	
Cooling and Heating Equipment		
III-(12)	6, 8, 21, 59,	92
IV-(12)	11, 12	73
Fluid Machine		
III-(13)		62
IV-(13)	13, 15	
III-(14) Food Products Machine		112
III-(15) Dyeing Machine	17	
III-(16) Printing Machine	1, 55	
III-(17) Plastic Moulder		94
III-(18) Other Transport Equipment	44	68
IV-(19) Fire Extinguisher	14	
IV-(20) Hand Tools		100
IV-(21) Household Equipment		117
IV-(22) Sewing Machine		72
IV-(23) Office Equipment		78

APPENDIX I

Calculations of the optimum solutions for the production activities for which $MC_1 < W_1$ and $\bar{X}_1 < B_1$ in Case D

Resource Element and Shops	1. Ball Mill $B_1 = 810$ H_{k1}	$H_{k1}B_1$	2. Coal Cutter $B_2 = 25.5$ H_{k2}	$H_{k2}B_2$	3. Portable Air Compressor $B_3 = 170$ H_{k3}	$H_{k3}B_3$	(a) $\sum_{1=1}^{3} H_{k1}B_1$	(b) [5] $U\left(\frac{\sum_{1=4}^{120} H_{k1}\hat{X}_1}{G_k}\right)G_k - \sum_{1=4}^{120} H_{k1}\hat{X}_1$	(c) (b) − (a)	F_k	G_k	(d) $F_k U\left(\frac{-(c)}{G_k}\right)$ if (c) < 0, 0 if (c) > 0
RE 2	0.0	0.0	0.00424	0.11	0.0	0.0				5300.8	6.0	
S 3		0.0		0.11		0.0	0.11	2.19	2.08			0.0
S 6		0.0		0.11		0.0	0.11	5.09	4.98			0.0
RE 3	0.00074	0.60	0.0	0.0	0.0	0.0				7861.4	4.0	
S 4		0.60		0.0		0.0	0.60	2.31	1.71			0.0
RE 8	0.0	0.0	0.00102	0.03	0.00059	0.10				2469.8	10.0	
S 2		0.0		0.0		0.10	0.10	0.0	−0.10			2469.8
S 4		0.0		0.03		0.0	0.03	3.91	3.88			0.0
RE 9	0.00167	1.35	0.0	0.0	0.0	0.0				8541.1	6.0	
S 5		1.35		0.0		0.0	1.35	4.23	2.88			0.0
RE 11	0.00044	0.36	0.00141	0.04	0.0	0.0				8383.9	8.0	
USS [1]		0.36		0.0		0.0	0.36	0.0	−0.36			8383.9
USS		0.0		0.04		0.0	0.04	0.0	−0.04			8383.9
RE 13	0.00030	0.24	0.00112	0.03	0.0	0.0				5015.4	8.0	
S 3		0.24		0.03		0.0	0.27	4.15	3.88			0.0
RE 14	0.0	0.0	0.0	0.0	0.00118	0.20				1490.1	12.0	
S 2		0.0		0.0		0.20	0.20	7.50	7.30			0.0
RE 15	0.00037	0.30	0.00118	0.03	0.00294	0.50				1784.4	12.5	
S 11		0.30		0.03		0.50	0.83	8.31	7.48			0.0
RE 16	0.0	0.0	0.00099	0.03	0.0	0.0				2956.8	10.0	
S 4		0.0		0.03		0.0	0.03	8.14	8.11			0.0
RE 19	0.0	0.0	0.0	0.0	0.00588	1.00				896.7	10.0	
S 7		0.0		0.0		1.00	1.00	7.74	6.74			0.0
RE 20	0.00030	0.24	0.0	0.0	0.0	0.0				25111.1	7.5	
S 2		0.24		0.0		0.0	0.24	7.18	6.94			0.0
RE 28	0.0	0.0	0.00376	0.10	0.00118	0.20				1718.9	10.0	
S 1		0.0		0.0		0.20	0.20	6.50	6.30			0.0
S 4		0.0		0.10		0.0	0.10	8.67	8.57			0.0
RE 29	0.00119	0.96	0.0	0.0	0.0	0.0				10241.8	6.0	
S 4		0.96		0.0		0.0	0.96	4.91	3.95			0.0
RE 32	0.0	0.0	0.0	0.0	0.00176	0.30				279.1	11.5	
S 5		0.0		0.0		0.30	0.30	0.0	−0.30			279.1
RE 33	0.0	0.0	0.00212	0.05	0.0	0.0				559.8	10.3	
USS		0.0		0.05		0.0	0.05	0.0	−0.05			559.8
RE 35	0.00074	0.60	0.0	0.0	0.0	0.0				1881.2	8.8	
S 1		0.60		0.0		0.0	0.60	7.03	6.43			0.0

	1. Ball Mill	2. Coal Cutter	3. Portable Air Compressor
(e) $\sum_k F_k U\left(\frac{-(c)}{G_k}\right)$	8383.9	8943.7	2748.9
$W_1 - MC_1$ [2]	103.0	93.96	42.69
(f) $(W_1 - MC_1)B_1$ [3]	83430	2396	7258
(g) (f) − (e) [4]	75046	-6548	4509
Optimum Level of Production: B_1 if (g) > 0, 0 if (g) < 0	810	0	170

[1]USS denotes "unshared shop"

[2]Marginal saving due to domestic production

[3]Total saving due to domestic production

[4]Net gain due to domestic production

[5]Excess capacity of the shop when production activities 1 = 4. ...120 are considered.

DRAFT: 29 November 1971
Not to be quoted without
authors' permission.

VRC
[illegible] file

A Model
for Evaluating Investment Projects in the
Mechanical Engineering Sector

by

Larry E. Westphal and Yung Whee Rhee*

Introduction

This paper discusses a static, one region model designed to aid in the selection of prospective projects in the mechanical engineering (ME) sector. The model is applied to planning investments in the ME sector (including electrical and non-electrical machinery and transport equipment, but excluding electronics) during the Republic of Korea's Third Five-Year Plan (TFYP). The advantage of our approach over traditional methods of project evaluation is the explicit recognition given to interdependence among different projects within the sector. Economies of scale and joint production, the dominant sources of interdependence, are specified in the model, which is formally a mixed integer programming problem.

---

* This paper is a report on research sponsored by the Development Research Center, International Bank for Reconstruction and Development (IBRD), which bears no responsibility for the opinions or conclusions expressed. The estimation of the model's parameters was under the direction of Mr. Nam, Joon Woo, Head, Equipment Engineering Laboratory, Korea Institute of Science and Technology (KIST), and was financed through contracts from the IBRD and the United States Agency for International Development's mission in Korea (USAID/Korea). For assistance in various phases of this project, we wish to thank: Mr. Cho, Sung Nak, Section Chief, Second Investment Division, Economic Planning Board of the Republic of Korea (EPB); Dr. Harry Yong Hwa Choi, former Senior Advisor, Battelle Memorial Institute; Dr. Louis Goreux, Director, Development Research Center, IBRD; Mr. Kim, Zae Kwan, Head, Special Equipment Laboratory, KIST; Dr. Lee, Kyung Suh, Head, Thermo-Hydraulic Laboratory, KIST; Mr. Lee, Hee Il, Director, Economic Planning Bureau, EPB; Mr. Thomas Olmsted, Assistant Director for Economics, USAID/Korea; and Dr. Hadley Smith, Industrial Economist, USAID/Korea. We owe a particularly heavy intellectual debt to Prof. Thomas Vietorisz whose work on planning in the mechanical engineering sector provided the jumping-off point for the work described here.

As presently implemented for the Korean case, the model focuses on choices between domestic production and imports; however, the model can easily be extended to include choices among alternative production methods for individual products. The objective of our planning exercise is to minimize the cost of meeting a fixed bill of final demands -- for new investment, replacement, consumption, and export -- for ME products. Since demand functions for ME products are infinitely inelastic while supply functions for inputs to the ME sector are infinitely elastic, the model does not adequately describe the interface between activity in the ME sector and in the rest of the economy. Given the difficulties of building a model for the ME sector alone, it appeared overly ambitious to extend the model's scope beyond the ME sector. Even so, the model's description of production activity in the ME sector is greatly simplified. Our model is highly experimental in that the motivation for its construction is to determine the potential utility of the programming approach to planning investments in the sector. Korean data are used to obtain concrete numerical results. The character of solutions to the model will determine whether or not our approach holds any promise.

The formulation and implementation of this model are a "joint venture" with a group of Korean mechanical engineers at the Korea Institute for Science and Technology (KIST). While primarily responsible for gathering technical information to implement the model, the KIST engineers have had a definite impact on its formulation through their acute awareness of the problems of estimation and their detailed knowledge of the sector. It is anticipated that, if successful, the model will be used in cooperation with Korea's Economic Planning Board (the government ministry responsible for planning) as a partial guide to project selection. Our model building effort has greatly benefited from several previous studies. The first /9/ is an extensive study of "initial conditions" and investment prospects in the Korean ME sector carried out by the same group of engineers under the direction of Dr. Harry Choi, formerly a Senior Advisor at Battelle Memorial Institute. The second /6/ is a study of the Soviet ME sector carried out by the Institute for Research in Social Science, The University of North Carolina, which first proposed the "resource element, shop concept" used here (with some modifications) to describe processing inputs. The third is the ongoing work of Prof. Thomas Vietorisz /13, 14, 15/, which is an outgrowth of the North Carolina study.

Aggregation Principles and Interpretation

At the beginning of our research it was expected that the model would focus exclusively on the set of prospective projects for the TFYP, incorporating existing production activity only where necessary to capture inter-relations affecting the costs or benefits of the proposed projects. The spirit of the planning exercise was to be that of a "complex analysis" (see /7/) in which the aim is to design a well integrated industrial complex. Depending upon the circumstances, the ME sector can be visualized as being a single industrial complex or as being comprised of a number of (perhaps related) industrial complexes. The advantage anticipated from the restricted scope of a complex analysis model was greater detail on the prospective projects and on the inter-relations among them; most important, by narrowing the focus of the model, we hoped to avoid aggregation over the major products. The engineers responsible for assembling technical data were at first sympathetic to this approach, but for several reasons quickly became disenchanted with it. First, a conceptual framework for modeling the entire sector was available from previous research elsewhere /6/, but there was no previously developed framework for modeling a sub-set of production activities. Second, it was felt that too much detail would jeopardize successfully estimating the technical parameters as well as the model's potential contribution to project selection (even given successful estimation). Finally, an impressionistic survey indicated that a major share of the sector's investment during the TFYP would go to the expansion of already existing plants. On the last ground, one might expect our model to specify, in very concrete detail, plant expansion activities for the already existing plants in a manner similar to that used in Kendrick's model of the Brazilian steel industry /8/. This is not the case; the model has very little detail on existing plants for the simple reason that the engineers felt it would be impossible to obtain valid technical data on existing capacity and proposed expansions from Korean producers.

In response to the engineers evolving preferences, the model as implemented in the Korean case is sector-wide in coverage and quite highly aggregated. The model is perhaps best understood as a tool of "pre-investment analysis" which can be used as a screening device to obtain an initial ranking of activities in the sector by comparative advantage preparatory to the design of specific projects. These projects would themselves require more detailed benefit-cost appraisal at a later stage. Our own preference as regards implementation remains with the "complex analysis" approach, for it avoids the conceptual ambiguities (noted below) of the present approach. As a tool of complex analysis, we expect the model would be primarily useful as a check upon the consistency of the over-all design of an investment program for the sector. Given the degree of interdependence characteristic of the sector, in designing individual projects within the sector, engineers must constantly make assumptions regarding activity in the rest of the sector. For example, the cost of a component will depend upon the number of end products in which it is used and the volume of production of each of these; the choice between subcontracting a part's production or producing it internally will depend upon production schedules in subcontracting shops; and so on. Designing an industrial complex for the ME sector is a simultaneous equations problem which must be approached piece-meal, project by project; once the complex is designed, our model could be used to check overall consistency and to modify the complex's design to achieve a more consistent organization. But this is not the mode of application here.

The model includes production and import activities for components, assemblies, and end products. The use of common components and sub-assemblies in higher order products is an important source of interdependence within the ME sector. Since virtually all ME products are produced under strong economies of scale, it is necessary to keep track of the demand for lower order products to calculate their production costs correctly. The distinction between components, sub-assemblies, and end products is rather arbitrary and is based on relative positions within the "component trees" of various end products. Many components and sub-assemblies are demanded for use outside the ME sector either as intermediate inputs or as replacement parts; but by definition, none of the end products goes into the assembly of other end products. In all, the model includes 120 different components, subassemblies, and end products, each of which requires a separate material balance constraint. The only other constraints appearing in the model are associated with the processing inputs. There is one production activity and an import activity for each product. The other activities in the model involve processing inputs. Several of the end products appearing in the model are aggregates over a number of individually specified items (e.g., machine tools); the composition of each aggregate is fixed in terms of each item's proportion in the total by physical weight. Production and import activities for these aggregated end products were obtained by the appropriate weighted averaging process. It is convenient to specify fixed proportions within aggregates where the technology is such that there is very little flexibility with respect to output composition.

Conceptually, each of the individually specified "products" appearing in the present model represents a whole class of products. For example, the product "fractional horsepower electric motor" designates the class of electric motors of less than one horsepower, rather than a particular motor. This being the case, the technical coefficients describing the production of fractional horsepower electric motors should be weighted averages of the coefficients pertaining to individually specified fractional horsepower electric motors, the weights being proportional to output shares. The data requirements to estimate the appropriate weighted average coefficients are immense, as separate estimates for each of the individual products are required. Furthermore, the interpretation of the model's results, which would be in terms of "weighted average products" rather than individual products, is unclear. We have been able to obtain production coefficients for only a single individually specified product within each of the different product classes that are not proper aggregates. The engineers working on the project preferred to have a single product serve as the "representative" of the whole class. Again, this makes the interpretation of the results very ambiguous. In the real world it is consistent to choose to produce some, but not all, of the domestically purchased fractional horsepower electric motors. The choice is posed in an all-or-nothing manner in the model.

The aggregation problem is one that plagues all attempts to build planning models and we fear we have nothing of substance to contribute to its resolution in this particular application. In future applications, we would argue that the model's scope should be restricted to that of a "complex analysis" in which each potentially producible, individually specified product appears separately without aggregation except where convenient and possible to apply proper aggregation principles. One would like to see rigorous criteria applied to the selection of the representative products, but we have been unable to direct substantial efforts in this direction. For the most part, we have chosen as representative products those that will have the largest shares either of the absolute output or the anticipated growth in output within their respective product classes. The engineers who estimated the technical coefficients in the model claim that the resulting coefficients are as close to the appropriate weighted average coefficients as one can come without estimating each component coefficient. In fact, they claimed that in many cases the most efficient procedure for estimating weighted average coefficients is through the use of representative products, the coefficients for the representative products being modified where necessary to obtain estimates of the weighted average coefficients. In some cases such adjustments were actually made in the data appearing in our model. We remain somewhat skeptical of the use of representative products to estimate weighted averages and note that, in any event, the interpretation of results for weighted averages is equally as unclear as the interpretation of results for representative products. Problems of interpretation reinforce the a priori notion that a model like that constructed here is really most useful (and perhaps only useful) where there is "man-machine" iteration between industry specialists and the model.

Classification of Inputs

Our discussion will be facilitated if a vocabulary is first established. A simple two-way division between inputs distinguishes between 1) intermediate inputs directly related to an item's production, and 2) process inputs. Direct intermediate inputs can be further broken down into lower order ME products and non-ME products, and within each of these categories there is a division into domestically producible and imported intermediates. Domestically producible inputs are those which are likely to be produced in Korea in 1976, leaving as imported inputs those that will most probably be imported in 1976. The former, excepting ME products, are valued at domestic producer's prices, while the latter are valued at c.i.f. import prices.

Requirements for domestically producible intermediate ME products are expressed in quantity (rather than value) input-output coefficients. Requirements for domestically producible non-ME products are expressed in value terms and enter the objective function as a cost of production. (However, we do have quantity input-output coefficients for 15 classes of raw materials, as the cost estimates were obtained by applying unit values to quantity input-output coefficients.) Requirements for imported ME and non-ME intermediate products are expressed in dollars and enter the objective function as costs of production after conversion to won.

Processing inputs account for labor, capital, and indirect materials costs, an example of the latter being fuel cost. Following the methodology initiated with the University of North Carolina's (UNC) study of the Soviet economy /6/, processing inputs are embodied in a number of "resource elements." A "resource element" is a generalized or representative processing facility specified at the shop, rather than plant, level. In the model's implementation, we have distinguished between 37 resource elements.
These are further broken down as follows:

Category of Resource Element	Number
Free Forging	5
Die Forging	2
Iron Casting Foundry	3
Steel Casting Foundry	2
Non-ferrous Casting	1
Die Casting	1
Heat Treatment	2
Machining	5
Surface Treating	1
Stamping	4
Upsetting	1
Fabrication	3
Assembly	7

Within each of the categories, individual resource elements are distinguished by the number of pieces of equipment, the maximum weight of the parts worked, the maximum force developed, hourly output capacity, and/or the number of employees. We have not followed the UNC practice of giving a detailed specification of the equipment in each resource element.

Our treatment of processing inputs departs from the UNC methodology in two important respects: first, intermediate inputs are directly related to the production of individual products rather than being associated with the resource elements; second, all processing input requirements are stated in terms of "processing hours" (shop hours of processing required). The advantage to associating intermediate inputs with products rather than resource elements should be obvious: better cost estimates are obtained. The measurement of processing inputs in terms of shop hours yields fairly precise input estimates without the necessity to sub-divide resource elements into a larger number of categories.1/

Estimation of Processing Facility Costs

Production or processing takes place in plants, but the "plant" concept is not a convenient vehicle for stating processing inputs because of the specificity of individual plants and the infinite variety of multi- and single-purpose production units found in the sector. The "resource element" concept is at the sub-plant level, and represents a vertical decomposition of plants into their component parts. It is convenient to categorize plant activities into two types: 1) production (or processing) activity directly related to the production of

---

1/ The capacity of a precessing facility cannot meaningfully be measured along a single dimension such as "tons of processed output." Markowitz and Rowe /1/, p. 316/ ennumerate the following characteristics as equally important dimensions along which capacity must be measured: geometric configuration of pieces worked, dimensions and weight of pieces worked, precision required, length of production run, and the hardness of the piece to be worked. The "shop time" concept serves as a convenient single dimension within which all of these dimensions may be encompassed.

individual items (e.g. stamping, casting, machining, forging, etc.); and 2) servicing activity needed to organize, supplement, and maintain production activity but only indirectly related to the production of individual items. Servicing activity can be further broken down into: 1) organizational (includes product design, production engineering, and cost control, the planning and programming of production, marketing, research and development, and administration); 2) maintenance and repair of production equipment; and 3) fixture production (fixtures are tools, dies, patterns and other items used in the production of individual parts and representing small scale specialized production equipment).

The specification of processing requirements in terms of resource elements is relatively unarbitrary and unambiguous as long as it is restricted to production activity. Servicing requirements are functionally determined at the plant level and can only be arbitrarily linked to the production of individual items. Furthermore, a large share of the servicing activity required for a given plant can be purchased from outside the plant; the sharing of a particular servicing activity among a large number of plants can serve to reduce costs substantially below their level were each plant required to provide its own servicing activity internally. Short of building the model at the plant level, there is no happy resolution to the problem of correctly specifying servicing activity costs. In implementing the model, we have arbitrarily included the costs of servicing activity in the cost of individual production resource elements on the, assumption that servicing would be provided within plants rather than purchased from a central pool. Our model is focused on the economies possible through horizontal integration among the production of parts entering into diverse products (termed "capacity sharing" below). It unfortunately does not contain a proper specification of the economies realizable through the vertical integration of different shops into a single plant ("overhead sharing"). These economies stem from the indivisibilities found in servicing activities.

Processing costs (except as noted below) enter the model's objective function in the form of annual capacity rentals derived in part from total construction investment cost annualized using a capital recovery factor. Construction investment cost figures have been obtained in the following detail:

a) Cost of direct production equipment in the resource element:

i) Dollar cost for imported equipment;

ii) Won cost for domestically produced equipment;

iii) Won cost for building and structures

b) Cost of providing servicing activities:

i) Dollar cost for imported equipment;

ii) Won cost for domestically produced equipment.

iii) Won cost for building and structures.

The annual capacity rental also includes annual average expenditures on labor, imported materials, and domestically producible materials incurred to operate the servicing facilities.

Labor and indirect production materials are processing inputs, so that it would be in keeping with our approach to relate labor and indirect material (other than that used in servicing) costs to the utilization of individual resource elements. For some reason, presumably ease of estimation, the KIST engineers have preferred to give total direct labor and indirect materials costs for individual products without breaking these down by resource element. As long as the possibility of multiple shift or overtime operation is neglected, there is no reason to prefer dis-aggregated figures. However, it seems clear from the outset that single shift operation for most resource elements will be sub-optimal, so that some accomodation of the model's structure to the form of the data will be required.

Interdependence -- The Focus of the Model

Significant interdependencies among production activities stem from the strong economies of scale in processing activity and are transmitted through the sharing of common parts (components and sub-assemblies) and of processing facilities among different items. Capacity is shared among different items' production in several forms. The most obvious form occurs when two or more products are produced in the same plant. A less obvious, but probably far more consequential, form of capacity sharing exists when dis-similar components or sub-assemblies used in different higher order products are produced in the same plant and are then shipped to other plants for further processing. Capacity sharing, in other words, can take an indirect form in which capacity is "shared" through the use of jointly produced but distinct parts used in different products. Seen in this light, the use of a common component or sub-assembly in a number of different products is a special case of capacity sharing. This holds true, of course, only if the shared component is produced in a single plant.

The multi-purpose nature of ME production facilities makes possible the lowering of costs through wide-spread capacity sharing in forms other than the use of common parts. Capacity sharing is likely to be an important factor where the volume of production is sufficiently small to preclude specialized processing facilities designed for high output volumes (e.g., transfer lines or highly specialized machine tools). Where the volume of production of individual items is large enough to warrant the consideration of specialized plants and equipment, there will naturally be a trade-off between cost reductions through capacity sharing and through specialization (which limits or precludes capacity sharing). The market in Korea (and her potential export market for most items) is small enough that the possibility of specialized processing facilities may be neglected. One must be careful not to overstate the dichotomy between capacity sharing on the one hand and specialized processing facilities on the other, for there is a spectrum between these extremes as is easily recognized in the case of machine tools producing a number of different items each requiring specialized fixtures to be used in combination with the machine. Doubling the output of a facility formerly operated at half capacity through producing an equal volume of a second item will increase total capital costs somewhat due to the requirement for specialized facilities. We should point out that the effect of capacity sharing on the cost of servicing activity is being neglected here.

Economies of Scale

There are at least four distinct scale economies characterizing the production technology within the ME sector and resulting in significant interdependence. It is well to distinguish these to properly understand the scope of our model. The indivisibility of production equipment leads to decreasing average costs. It is not possible to obtain machine tools and other equipment in continuously varying capacities. Up to the capacity of a particular piece of equipment, the more it is used, the lower the unit cost per part produced (or per hour of utilization). While the appropriate specification of the indivisibility of individual pieces of equipment is quite straightforward, its correct extension to the case of shops is much less clear. It is to be expected that, depending upon the specific processing requirements of the various items produced, capacity utilization rates (measured in hours of use) for the individual pieces of equipment within a shop will be unequal. If some equipment is being fully utilized while the remaining equipment in the shop is under-utilized, does one say that there is excess capacity in the shop or that there is none? This is really a problem of properly defining a shop's capacity, and it has been more or less sidestepped here.

The indivisibility of the specialized fixtures used in the production of specific parts also leads to decreasing costs. Decreasing costs through higher utilization of generalized equipment requires a somewhat different specification than that appropriate in the case of fixtures. In the former case it is necessary only to know the utilization rate, regardless of the specific items being produced; in the latter case, it is necessary to keep track of the specific items being produced.

The third source of decreasing costs is found in the capital cost of production facilities. Though there is very little evidence, it is widely believed that equipment of greater capacity (measured here in terms of the actual processing carried out) costs less per unit of processing. The paucity of evidence on this point doubtless can be traced to the conceptual problems of measuring capacity and to the fact that very different types of equipment are used at different scales of output. At low to medium scales of output for a particular item, a given processing step is likely to be performed on a generalized machine that is shared with other items' production. At a very high scale of output, the same processing step might be carried out on a specialized machine. The prevalence of capacity sharing makes it difficult to determine scale economies in terms of the output of a single item, while the absence of an adequate concept of capacity precludes a measurement in terms of a more generalized capacity figure. Nonetheless, the observation that different equipment is used at different output scales leads to the conclusion that there must be significant economies of scale in the provision of capacity.[1/]

The final manifestation of economies of scale is found in the set-up time required to ready a machine for a particular production step. In the developed industrial countries, only 20 to 40 percent of the total production time using machine tools, for example, is consumed in the machining process itself /12, p. 451/. The rest of the time is spent in controlling the machine tool, set-ups and tear-downs, and measuring. The distinction between the fixed and variable processing times required to produce an individual item leads naturally to scale economies.

We have focused our attention on the indivisibility of processing facilities and on the economies of scale in their capital cost, feeling these to be the most significant sources of decreasing costs in the Korean context. By assuming that all fixtures are produced within the shop, we have included an aspect of fixture indivisibility. However, the use of fixtures is not functionally related to the production of individual items. Given the large number of products distinguished in the model, and the fact that not all the parts for these products are separately enumerated, we decided to neglect the distinction between fixed and variable processing times. The "processing hour" input coefficients appearing in the model include the average (with respect to anticipated scale of output) time spent on control, set-ups and tear-downs, and measuring per item. Significant economies of scale are also found in the provision

---

[1/] In the one empirical investigation of which I am aware, Alpert /1, p. 180/ has found that the scale elasticity of plant cost in the production of turbine engines is 0.25 (Alpert uses tons of processed output as the measure of capacity). When compared to estimates in the neighborhood of 0.7 for the process industries, scale economies in the ME sector are significant indeed.

of servicing activity. We have neglected these, except insofar as they are reflected at the shop level, in keeping with our focus on horizontal rather than vertical integration. The appropriate vertical integration of shops into plants is likely to depend upon a number of organizational factors that cannot, at this stage, be included in a programming model of the character we are concerned with.

Variable Capacity Resource Elements

Given that production facilities in the ME sector have been compressed into 37 resource elements to simplify the implementation of the model, the assumption of fixed scale or fixed capacity for individual resource elements is not very appealing. Assuming single shift operation, each resource element has a capacity of 2,000 processing hours. Production of some of the individual products at the anticipated output volume would require more than 20,000 processing hours in particular resource elements; for other individual products the total processing requirement for a particular resource element may be as small as 50 hours. In the first case, more than ten shops within a resource element category would be needed to produce various parts for one item were processing facilities specified in terms of fixed capacity, indivisible resource elements. But in reality, at most two or three shops of larger scale would be used; within the Korean context it is likely that only a single shop would be licensed. What is needed is a way of incorporating this into our model through variable capacity resource elements. The reader is cautioned that there is a serious ambiguity here. We are assuming that the production of given parts for a certain product will be carried out in the same class of resource element regardless of the scale of production. But this is not always true; At one volume of output for a particular product, the requirement may be for a forging shop falling within one resource element category, while the requirement at another volume of output may be for another forging shop falling in a different resource element category, or even for a casting, machining, stamping or fabrication resource element. It would be possible to specify a model reflecting this fact; but we did not have the resources to do so.

The measurement of both processing requirements and capacity in processing hours makes it difficult to specify variable capacity resource elements. Regardless of its capacity measured along other dimensions, any shop has 2,000 processing hours capacity (continuing to assume single shift operation). Two modifications of the "processing hour" concept have been used to estimate capacity cost functions on the assumption of variable scale.

1. Machine Time: The initial measurement of processing inputs is in terms of hours of shop time, and the resource element is defined in terms of the number of pieces of equipment contained. Variable scale can be accomodated if capacity and production requirements are stated in machine, and not shop, time. The machine time capacity of a particular shop is simply equal to the number of machines it contains multiplied by the number of hours in a year that the shop can be operated (i.e. its shop time capacity). Similarly, shop-time input coefficients can be converted to machine-time coefficients through multiplying by the number of machines in the shop assumed in the original definition of the resource element class. The implicit assumption in this concept is that larger capacity shops have more of the same kind of equipment, which need not in fact be true.

2. Prime Machine Time: This modification is similar to that discussed above except that only certain machines within individual shops are included to determine prime machine time capacity. Prime machines are those pieces of

equipment fundamental to the production process and around which any shop within the resource element class is designed; they are the machines most likely to yield the capacity limitations of the shop as a whole. For example, prime machines in the case of free forging shops might be the hammers or presses, while for die forges the prime machine is a drop hammer.

Capacity cost functions have been obtained under one or the other method by linearly interpolating between two point estimates, the second being for a shop of twice the initially assumed capacity. The linearly interpolated cost function is of the "fixed charge" type, with a positive intercept and a positive slope.

Specification of Capacity Sharing

One last task remains. Having specified economies of scale in processing, we must be careful to obtain correct estimates of output scale. Subject to the reservation that resource elements may have an upper limit to their capacity, the inclusion of all requirements for a particular resource element in a single constraint would involve the implicit assumption that all the processing falling in this class is performed in a single huge shop. This assumption is clearly unwarranted. In the first place, to spread a single servicing facility over so large a volume of production could well be inefficient. In the second place, our resource elements designate classes of shops. Within a particular class there is a wide variety of particularized shops, the design of each depending upon the mix of items being produced. While it is impractical to estimate separate cost and utilization parameters for each of the shops within a class, it would be a serious error to neglect completely the differences among these shops. We have gone part way toward recognizing the differences by separately estimating the processing requirements from each of the shops within a resource element category. This is accomplished by indicating those products whose requirements for processing in a particular resource element are likely to be performed in the same shop. A separate capacity utilization constraint is required for each shop so defined. In this way, more accurate estimates of the scale of processing activities are obtained.

The specification of capacity sharing permits consideration of the pattern of horizontal integration among activities within the sector. To go completely to the plant approach would require the identification of patterns of vertical integration among shops. The identification of shops is a row (resource element) wise listing of the columns (production activities) found together and having a common capacity constraint. The identification of plants would require, in addition, a column (production activity) wise listing of the rows (formerly resource elements, but now shops) found together in a common plant and sharing the same servicing facilities. We have not identified patterns of vertical organization.

A major objection to our approach should be recognized. There is a certain amount of vertical integration already present in the resource element concept. For example, it is likely that the machining (within a single resource element) of the many parts required for a farm tractor will be done in a number of different shops, and not in a single shop within the resource element class. Our approach assumes that all machining falling in a single resource element category, of parts for a tractor will take place in the same shop. The only justification we can offer is that the failure to disaggregate in this case is an unimportant source of mis-specification in a model for a country whose market is as limited as Korea's.

Statement of the Model

As a consequence of excluding alternative production activities for each product, the formulation of the model given below assumes that there is no substitution possible among shops within the same (or different) resource element in the production of a given item. At the cost of additional computational expense to solve the model, this assumption can easily be relaxed; it is made purely for expositional ease. (In fact, the model for Korea does contain limited substitution possibilities among shops within given resource elements.) The parameters and variables appearing in the model are defined below, following the convention that parameters are designated by upper case Roman letters, variables by lower case Roman letters.

Parameter	Definition
F_i	the annualized fixed cost component of the cost of a shop in the i'th resource element class
V_i	the annualized variable cost component of the cost of a shop in the i'th resource element class
C_l	the non-processing cost per M/T (metric ton) of producing the l'th item; includes the cost of domestically produced non-ME intermediate inputs, the cost of imported ME and non-ME intermediate inputs, and the cost of direct labor input
W_l	the c.i.f. import price of the l'th item
$H_{ij,l}$	the number of shop hours of processing required from shop j in resource element i per M/T of production of the l'th item
G_i	the upper bound to the number of shop hours (or capacity) in a shop in the i'th resource element class
A_{ln}	M/T of input of the l'th domestically producible item required per M/T of production of the n'th item
D_l	the exogenously given final demand (in M/T) for the l'th item

Variable	Definition
Δ_{ij}	integer variable associated with the j'th shop in the i'th resource element class
h_{ij}	total number of shop hours of processing required from the j'th shop in the i'th resource element class
x_l	volume of production of the l'th item, in M/T
m_l	volume of imports of the l'th item, in M/T

The use of the word "shop" in defining Δ_{ij} and h_{ij}, while convenient, is misleading. Since there is an upper bound to the capacity in shop hours of any shop within each resource element, the processing requirement from a shop of type j in resource element class i may require the construction of more than a single shop (i.e. $\Delta_{ij} > 1$). For want of a better word, perhaps "supra-shop" should be used in place of "shop" in the definition of Δ_{ij} and h_{ij}. A "supra-shop" is then designated by the collection of items that could be produced together in shops of resource element type i. The important point is that there is a limit to the extent to which economies of scale can be achieved through capacity sharing within a resource element class.

The term "shop hours" as used in the definition of $H_{ij,l}$, G_i, and h_{ij} refers to "effective shop hours" as measured in reference to a shop having 2000 working hours on a single shift basis. A shop having N times the capacity of the reference shop, measured either in machine or prime machine hours, has an effective shop hour capacity of 2000 x N shop hours. All shop hour magnitudes are measured in effective shop hours, and thus in relation to reference shops of 2000 shop hours. In the statement of the model below, we have assumed single shift operation. This assumption too can easily be replaced.

The statement of the model is given below:

Objective Function:

$$(1) \quad \min \sum_{i=1}^{37}\left(F_i \sum_{j=1}^{R_i} \Delta_{ij}\right) + \sum_{i=1}^{37}\left(V_i \sum_{j=1}^{R_i} h_{ij}\right) + \sum_{l=1}^{120} C_l\, x_l + \sum_{l=1}^{120} W_l\, m_l$$

where R_i = number of supra-shops in resource element class i

Constraints:

$$(2) \quad h_{ij} = \sum_{l=1}^{120} H_{ij,l}\, x_l, \quad i = 1,\ldots,37; \quad j = 1,\ldots,R_i$$

where at most one $H_{ij,l} > 0$ for given i and l

$$(3) \quad \Delta_{ij} = U\,(h_{ij}/G_i), \quad i = 1,\ldots,37; \quad j = 1,\ldots,R_i$$

where U (y) is a function giving the smallest integer larger than y

$$(4) \quad x_l + m_l - \sum_{n=1}^{120} A_{ln}\, x_n = D_l, \quad l = 1,\ldots,120.$$

Constraints (2) and (3) are required for the proper accounting of processing costs; constraint (4) is a standard material balance constraint. It should be noted that in determining the lowest cost means of satisfying Korea's 1976 demands for ME products, the model neglects existing capacity.

Each of the cost components F_i, V_i, and C_l can be disaggregated into the cost of domestically produced intermediate and capital inputs, the cost of imported intermediate and capital inputs, and the cost of labor. One can thus solve the model under different assumptions regarding the exchange rate and the hourly labor wage. The initial values for these parameters are respectively 300 won per US$ and 150 won per man hour. One can also solve the model under different assumptions regarding the interest rate, which affects the capital recovery factor used to annualize capital costs. Letting R denote the interest rate and N the expected life of a shop, the capital recovery factor is $\left\{R(1+R)^{1/2}/\left[(1+R)^{N+1} - (1+R)\right] + R\right\}$, where it is assumed that contributions to the depreciation reserve are made at mid-year.

Solution

It is convenient to re-express equations (1), (2), and (3) employing a slightly different indexing for the various shops. Let

$k = \sum_{s=0}^{i-1} R_s + j$ (where $R_0 = 0$), so that k is the index corresponding

to the j'th shop of the i'th resource element. Then define $\Delta_k = \Delta_{ij}$, $h_k = h_{ij}$, and $H_{kl} = H_{ij,l}$; and $F_k = F_i$, $V_k = V_i$, and $G_k = G_i$ where $\sum_{s=0}^{i-1} R_s < k \leq \sum_{s=0}^{i} R_s$. Equations (1) through (3) then become:

Fixed cost

$$(1') \quad \min \sum_{k=1}^{R^*} F_k \Delta_k + \sum_{k=1}^{R^*} V_k h_k + \sum_{l=1}^{120} C_l x_l + \sum_{l=1}^{120} W_l m_l,$$

$$(2') \quad h_k = \sum_{l=1}^{120} H_{kl} x_l,$$

$$(3') \quad \Delta_k = U(h_k/G_k), \text{ where } R^* = \sum_{i=1}^{37} R_i.$$

Upon substitution for h_k and Δ_k in (1'), the problem is

Objective F

$$(5) \quad \min \sum_{k=1}^{R^*} \left\{ F_k\, U\left[\sum_{l=1}^{120} (H_{kl}/G_k)\, x_l \right] \right\} + \sum_{l=1}^{120} \left\{ (C_l + \sum_{k=1}^{R^*} V_k H_{kl})\, x_l \right\} + \sum_{l=1}^{120} W_l m_l$$

subject to

$$(4) \quad x_l + m_l - \sum_{n=1}^{120} A_{ln} x_n = D_l, \quad l = 1, \ldots, 120.$$

If one neglects the fixed charges associated with shop inputs, then it can be shown that the unit cost of producing the l'th item is

$$(6) \quad MC_l = C_l + \sum_{k=1}^{R^*} V_k H_{kl} + \sum_{n=1}^{120} A_{nl} p_n, \text{ where}$$

$$(7) \quad p_n = \min (MC_n, W_n).$$

This unit cost concept has been designated "MC" for it yields the marginal cost (i.e. exclusive of fixed charges) of producing a unit. Clearly, MC_l must be less than W_l if the l'th item is to be produced in the optimal solution. If not a characteristic of the ME sector at large, it is nonetheless a characteristic of the Korean data that there is no circular interdependence among intermediate input flows. That is, it is possible to order production activities and products in such a way that $A_{nl} = 0$ for $n > l$; the matrix of intermediate input coefficients is upper triangular. One can thus determine each item's marginal cost recursively, starting with the first item and working through to the last. This results in substantial computational savings.

The first step of the solution procedure is to calculate the marginal cost of producing each item and set $\hat{x}_l = 0$ for items such that $MC_l > W_l$, where $\hat{x}_l$ denotes the optimal value of x_l. For plausible values of the foreign exchange rate, interest rate, and hourly wage, more than fifty per cent of the items are known to be optimally imported on this ground alone. Even so, one is left with a mixed integer programming problem having literally hundreds of integer variables if expressed using zero - one variables (recall that Δ_k can take on any integer value, though upper bounds can be determined). Given the expense of calculating solutions to mixed integer programming problems, alternatives must be sought to straightforward mixed integer programming. One such alternative has been employed in our computations to date.

We neglect the possibility of capacity sharing and calculate production cost when each item is produced in specialized shops producing only the single item.

The total cost of producing x_l metric ton of the l'th item in "unshared" shops when producible intermediate inputs are priced at the lower of the marginal cost and world market price is

$$(8) \quad \sum_{k=1}^{R^*} \left\{ F_k \, U \, (H_{kl} \, x_l \, / \, G_k) \right\} + MC_l \, x_l .$$

The rationale behind the pricing domestically supplied intermediate inputs at marginal rather than average cost will become apparent below. The choice between importing and domestically producing an item turns on whether production cost exceeds or is less than import cost. An upper bound on possible production volume is needed to compare production and import cost. Let us designate the upper bound to be B_1; the choice of the numerical value for B_1 is discussed below. Here we require only that it be less than the realized total final and intermediate demand for the 1'th item. Now the make-buy choice can be posed in terms of the following programming problem:

$$(9) \quad \min_{x_1 \leq B_1} \sum_{k=1}^{R^*} \left\{ F_k \; U \; (H_{k1} \, x_1 \; / \; G_k) \right\} - (W_1 - MC_1) \, x_1 .$$

The expression $(W_1 - MC_1)$ is the marginal savings over import cost associated with domestic production. It is quite simple to determine the minimizing value of x_1, for we need only evaluate the minimand for values of x_1 equal to zero and those marginally less than multiples of $G_k \; / \; H_{k1}$ (all k such that $H_{k1} > 0$) and less than B_1. Let $\bar{x}_1$ denote the optimal value of x_1 with respect to this problem. Though it need not be so, in all cases which we have investigated, $\bar{x}_1$ equals either 0 or B_1. Of course, we compute $\bar{x}_1$ only for those items for which marginal cost is less than import price.

If $\bar{x}_1 = B_1$ for all items for which $MC_1 < W_1$, then it can be shown that $x_1 = \bar{x}_1$ for 1 such that $MC_1 < W_1$ and $x_1 = 0$ for 1 such that $MC_1 \geq W_1$ is the optimal solution to the problem as posed in (4) and (5) above when the additional "complex scale constraints"

$$(10) \quad x_1 \leq B_1 \; , \; 1 = 1, \; \ldots, \; 120,$$

are included in the problem. Should some values of $\bar{x}_1$ be less than B_1, then the optimal solution value to the problem given by (4), (5) and (10) is merely bounded from above by the solution value for which $x_1 = \bar{x}_1$ for 1 such that $MC_1 < W_1$ and $x_1 = 0$ for 1 such that $MC_1 \geq W_1$. If the number of items for which $\bar{x}_1 < B_1$ is small, then it turns out that one can "finish off" the solution very quickly by hand calculations. We will not go into the mechanics of this process here, but will do so in the more complete exposition.

The parameters B_1 employed above can be rationalized on one of two grounds. The first places the planning exercise in the "complex analysis" context. We visualize the exercise starting off with engineers designing an industrial complex on the basis of a sequential, item-by-item evaluation and production plan. As each item is considered, assumptions are required regarding the scale of output of other producible items in order to price intermediate and processing inputs. At the conclusion of this first step, the engineers have designed a complex but the internal consistency of the assumptions made in the process remains to be demonstrated. It is here that our model is used to evaluate the overall production plan in a framework that focuses on interdependence. The B_1 parameters are simply the output scales given in the specification of the industrial complex designed by the engineers. The planning exercise would continue with "man-machine" iteration between model solutions and redesign by the engineers. Note that within this context final demand does not enter the model formally and is relevant only at the complex design stage.

The second rationalization of the B_1 parameters brings final demand formally back into the model and, at the same time, makes solution potentially far more difficult. An upper bound to the total demand for any item is given by

$$D_1 + \sum_{n \in n^*} A_{1n} x_n \ ,$$

where n^* is the set of items for which $MC_n < W_n$ and one solves for demand values successively in the reverse order to that used in computing marginal cost values. The initial values of the B_1 parameters are set equal to the upper bounds just determined and one solves the model as indicated above. Then the B_1 values are recalculated on the basis of the initial solution and one proceeds iteratively until the assumed B_1 values are consistent with the implied solution values for the domestic production of each item. We have not followed this procedure, nor have we investigated its convergence properties. Our method is consistent with the first interpretation. In fact, up to this point, we have not formally estimated final demands; instead we have exclusively considered the optimality of the output scales assumed by the KIST engineers in estimating the coefficients of the model.

While it is extremely fortuitous that the numerical values of the parameters in our model permit solution following the rather simple method outlined above, it is also discomforting, for it means that interdependence through capacity sharing rarely "matters." Whether this conclusion needs to be tested against the original model in which the output bounds given by the engineers' design are dropped is <u>the</u> issue for consideration in the seminar.

BIBLIOGRAPHY

1. Alpert, S. B., "Economies of Scale in the Metal Removal Industry," Journal of Industrial Economics, July 1959, 175-81.

2. Baranson, J., Manufacturing Problems in India: The Cummins Deisel Experience, Syracuse: Syracuse University Press, 1967.

3. Boon, G. K., Economic Choice of Human and Physical Factors in Production, Amsterdam: North-Holland Publishing Company, 1964.

4. Gallik, D., "Explorations in the Development of Pre-Investment Data for the Mechanical Transformation Sector," United Nations Center for Industrial Development consulting report, IDP/EWG.6, 1961. (Mimeographed.)

5. Institute for Research in Social Science, "Input-Output Analysis of Soviet Heavy Machinery (Soviet Planning Study No. 6)," Chapel Hill: Institute for Research in Social Science, University of North Carolina, 1958. (Mimeographed.)

6. Institute for Research in Social Science, "Profuction Coefficients and Technological Trends in Soviet Industry: An Input-Output Analysis of Machinery Production (Soviet Planning Study No. 7)," Chapel Hill: Institute for Research in Social Science, University of North Carolina, 1959. (Mimeographed.)

7. Isard, Walter, et. al., Industrial Complex Analysis and Regional Development, Cambridge: The M.I.T. Press, 1959.

8. Kendrick, D. A., Programming Investment in the Process Industries: An Approach to Sectoral Planning, Cambridge: The M.I.T. Press, 1967.

9. Korea Institute of Science and Technology, Plan for Development of Korean Mechanical Engineering Industry, Seoul, Korea: Korea Institute of Science and Technology, 1970.

10. Korea Institute of Science and Technology, Final Report on Development of a Technology Matrix for the Mechanical Engineering Sector, Seoul, Korea: Korea Institute of Science and Technology, 1970.

11. Markowitz, H. M., and A. J. Rowe, "A Machine Tool Substitution Analysis," Studies in Process Analysis, edited by A. S. Manne and H. M. Markowitz, New York: John Wiley and Sons, Inc., 1963.

12. UNIDO, Report of the Interregional Symposium on Metalworking Industries in Developing Countries, Vienna: UNIDO, ID/8, 1968.

13. Vietorisz, T., "Alternative Approaches to Metalworking Process Analysis," Studies in Process Analysis, edited by A. S. Manne and H. M. Markowitz, New York: John Wiley and Sons, Inc., 1963.

14. Vietorisz, T., "The Planning of Production and Exports in the Metalworking Industries," Vienna: UNIDO, ID/WG.10/1, 1969. (Mimeographed.)

15. Vietorisz, T., "Programming of Production and Exports for Metalworking: Models and Procedures," Vienna: UNIDO, ID/WG.10/2, 1969. (Mimeographed.)

Memorandum 71-

November 1971

Gary P. Kutcher*

DRC

DEC 14 1971

file

DECOMPOSING PRICE-ENDOGENOUS PLANNING MODELS:

PRACTICAL CONSIDERATIONS AND THEORETICAL IMPLICATIONS

Contents Page

---

* This will form a chapter in the forthcoming book "Project Decisions and Multi-Level Planning: Case Studies in Mexico".

The author is grateful to Roger D. Norton for initial stimulation in this area. In addition, Clopper Almon, John H. Duloy, Louis M. Goreux, Alan S. Manne, and Janos Kornai provided helpful discussions.

1. Decomposition and the Mexico project

In the late nineteen-fifties and early' sixties, when economists and other linear programming model builders had not yet learned to live within the bounds of current computational technology (as if they have done so to date!), several algorithms for decomposing large programming problems were devised, most notable being that of Dantzig and Wolfe (1961). The basic premise of decomposition algorithms is simple: if a computer can easily solve relatively small problems but not very large ones, perhaps a large problem can be broken up into several smaller ones, linked together in such a way that their ultimate solutions, or a combination or representation of their solutions, provide a solution to the full problem.

Since economists were involved in the evolution of decomposition techniques (as they have been involved in mathematical programming in general) from the start, an analogy between decomposition and economic theory was provided quickly. The component problems (sub-programs) can correspond to individual decision-making units, each with its own objectives and resources, and the linkage between the sub-programs can represent an "invisible hand". Or, if the linkage is a master problem, it can be considered as the "planning bureau", if one's political philosophy does not object. Because of these analogies, interest in decomposition was inspired from two different sources: the desire to solve problems which, in their original state, would exceed existing computational capacity, and from the ability of a decomposition approach to partially simulate actual market and planning processes involving price-motivated behavior, production quotas and/or resource endowments.

An example of the purely computational interest is a model of an oil company comprised of several producing fields. Because the bulk of the constraints are specific to the fields, the problem can be decomposed such that the fields are submodels, and the tasks of the full problem are reduced to allocating the common resources and/or inducing each field to produce that level of output which optimizes the full problem.

The classic example of the economist's interest is that of an economy-wide model comprised of various sectors. Again many of the constraints in such a model are likely to be specific to the sectors (e.g., specialized types of labor and physical capital) while other common resources must be allocated among sectors, usually by finding a set of prices such that the marginal contribution to the economy's objective is the same for all sectors for any given resource. In such constructs there are two levels of decision-making: the economy-wide and the sectoral. The tasks of the higher level are typically to coordinate the production decisions of the sectors so as to achieve some overall goal or optimum (e.g., meet output targets at minimum cost or maximize some measure of social welfare), while the duties of the sectors are to conform to the directives of the higher level (center) either directly by meeting quotas or indirectly by reacting optimally to price signals or resource allocations. Under certain assumptions the beauty of competitive equilibrium-seeking dynamics can be simulated by a decomposition algorithm: if prices are endogenous and the submodels represent consuming and price-taking producing units, obtaining a solution to the full model amounts to finding a set of prices such that the usual competitive equilibrium conditions are met.

Of course not all linear programming problems lend themselves to efficient solution by decomposition; only those characterized by block-diagonality.

Such a problem is illustrated schematically in Figure 1 for n + 1 partitions.

Several distinct decomposition algorithms have been designed to solve problems of this structure, the most well-known and most applied being the Dantzig-Wolfe. According to Orchard Hays, perhaps the most experienced practitioner of decomposition techniques, it is unequaled for elegance and versatility.*

In practice, application of decomposition algorithms have presented difficulties. Most problems do not fit neatly into the structure of Figure 1; obtaining satisfactory partitioning can be tedious and require obscuring the logical structure of the problem; "callable" linear programming computer routines sufficiently powerful to handle the subprograms and master (if any) are difficult to come by; problems connected with finding initial basic solutions (especially to the master) and dealing with degeneracy in sub-models must inevitably be handled as special cases. For these reasons, and because powerful "canned" linear programming routines are now widely available, interest in decomposition has waned of late. In the words of Orchard-Hays,

> "Though perhaps inevitable, this is unfortunate since decomposition is the only really promising extension to mathematical programming for large and complicated models." *

At the outset of the project involving the models described in this monograph, a decomposable system involving at least three levels of decision-making was envisaged: the economy-wide, the sectoral, and the agricultural district. DINAMICO, the economy-wide model, is comprised of sixteen sectors. One such sector, energy, is represented by ENERGETICOS. In CHAC, the model of the agricultural sector, production is segregated by districts. Even within these districts, production decisions could be broken down by land class or farm type, as in the highly detailed model of the Bajio.

---

* Orchard-Hays (p. 240)

Although information gained from simulation of the linkages among these decision levels by decomposition could be enormous (for a sample see the section DINAMICO-CHAC linkages), the original intent of a decomposition system has not been carried through to date primarily because of exigencies connected with early completion of the "first round" of the project and the availability of computer routines which could solve the largest and most complex component, CHAC.

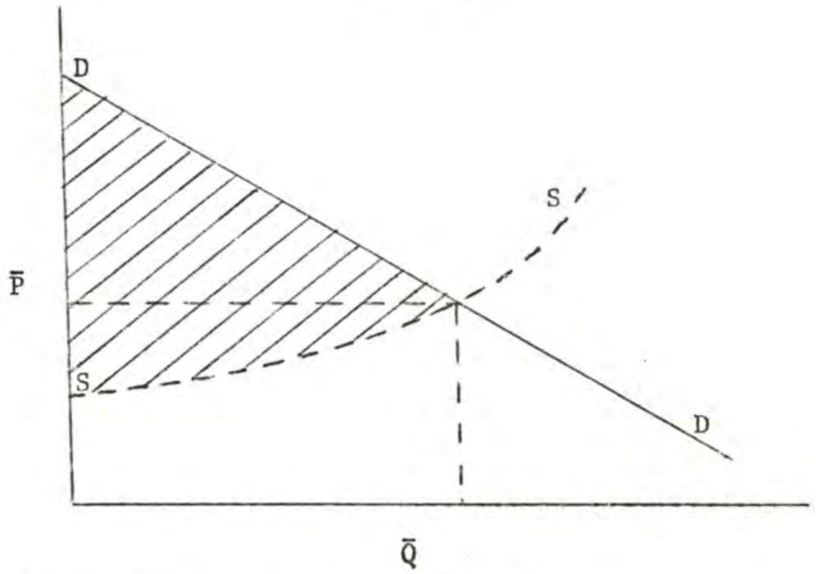
Efforts to decompose CHAC, however, proceeded independently largely out of a desire to make the model operational in Mexico for use in current policy decisions. (Computer technology in LDC's often lags a generation; as of this writing only the aggregated version, CHAQUITA, has been solved on the equipment available to our collaboraters in Mexico). A description of the efforts toward decomposing CHAC using the Dantzig-Wolfe algorithm, and the lessons derived therefrom comprise the remainder of this chapter.

The Dantzig-Wolfe algorithm was selected as a starting point, largely out of familiarity, but also because its reliance on price signals seemed to offer greater promise for efficient solution of the price-endogenous agricultural models than other quota-oriented algorithms. Before proceeding with a description of the experiments, we make two detours: one to review the basic structure of CHAC with emphasis on those aspects which could influence the efficiency of a decomposition procedure, and secondly, to review the steps of the D-W algorithm.

1.1 CHAC: An overview

The salient characteristics of CHAC, the model of the Mexican agricultural sector, derive from its objective function, the maximization of consumer and producer surpluses. Recall that the objective function entries in the CHAC demand activities account for the area under the demand curve DD in Figure 2 while negative objective function entries in factor pricing activities subtract the area under the supply curve, SS.

Figure 2

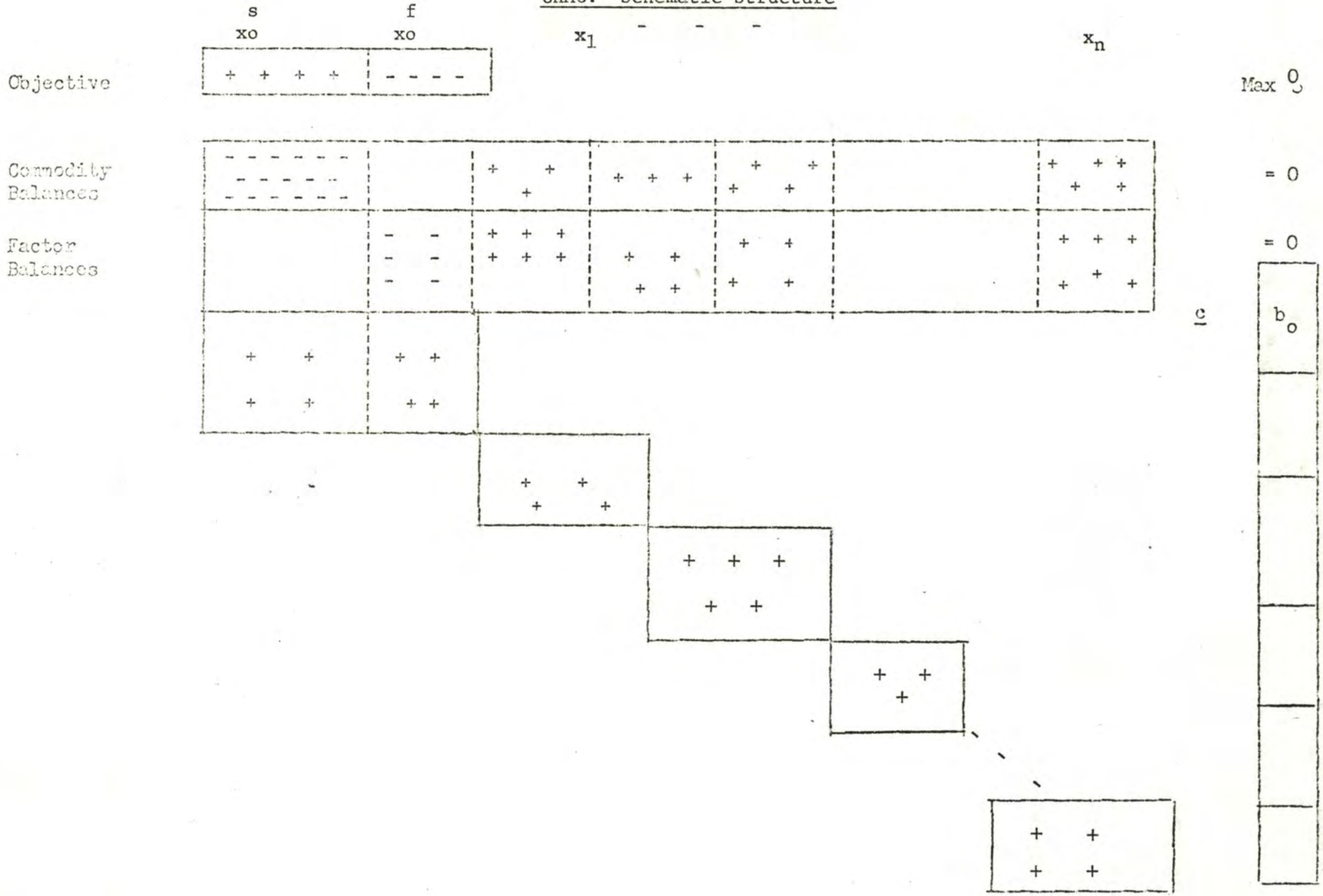


In the figure, the supply function is drawn as a dotted curve because of the inherent interdependence (competition for resources and markets) involved. Optimization of the model leads to a competitive solution in the sense that price equals marginal cost. Figure 3 is a schematic representation of the full model, and illustrates the three broad categories of activities: demand, factor supply, and production. The constraints can also be classified into three categories: commodity balances (positive entries in production activities; negative in demand sector-wide factor balances and constraints, and producing district

Of the activities in CHAC, are demand or demand-related, are related to factor supply, and are production or district-specific factor supplying or transport cost charging activities. Of the constraints, there are commodity balances, connected with constraining or charging sector-wide factors, and constraints which are district-level.

Figure 3

CHAC: Schematic Structure



The remaining rows are required for accounting various measures of income, employment, and production. From the above tabulation, it is clear that CHAC is highly amenable to solution by decomposition as the structure is inherently block-diagonal.

Besides CHAC, two other price endogenous agricultural models have evolved in the course of the project. They are the aforementioned CHAQUITA, which is virtually identical to CHAC except that the monthly constraints on labor, land, and irrigation water have been aggregated into three "seasons", and PACIFICO, a model of the Pacific Northwest region comprised of modified versions of five irrigated district models and regional demand functions.

In addition, a purely experimental model, POQUITA, was devised for purposes of initial decomposition experiments. In the experiments described later, only results from POQUITA and PACIFICO will be presented.

1.2 Dantzig-Wolfe: The rudiments

The algorithm devised by Dantzig and Wolfe is based on the idea that a solution to the complete problem can be obtained as a combination of solutions to the subproblems (i.e., solutions constrained only by the resources specific to each of the subproblems). The common constraints are met by devising varying objective functions for the submodels which reflect the evaluation of the common constraints according to a master problem. At some point, the solution of the master problem which is comprised of representations of the subproblem solutions, gives the solution of the full problem. Thus the algorithm consists of obtaining subproblem solutions, sending representations of these solutions to the master problem, solving the master, revising the objective function entries of the subproblems on the basis of the master's evaluation of the common constraints, obtaining new subproblem solutions, etc., until some optimality test is passed.

More precisely, the steps of one version of the algorithm are:

1. maximize $c'_j x_j = (c_j - \pi A_j) x_j$

   subject to $x_j \geq 0$, $B_j x_j \leq b_j$

   (where π is the vector of shadow prices on the common constraints.)

2. calculate $P_{jk} = A_j x_j$

   and $c_{jk} = c_j x_j$

   (the k subscript refers to the iteration (cycle)).

3. solve the master problem consisting of finding activity levels s_{jk} such that

   $$\sum_{j}\sum_{k} P_{jk} s_{jk} \leq b, \; s_{jk} \geq 0 \qquad \sum_{k} s_{jk} \leq 1$$

   which maximize $\sum_{j}\sum_{k} c_{jk} s_{jk}$

If none of the current P_{jk} enter the basis of the master, the previous solution of the master provides the overall optimum. Otherwise,

4. calculate

   $$c'_j = c_j - \pi A_j$$

   and return to (1).

The above, of course, departs slightly from the original D-W formulation. Aside from our consideration of a maximization problem rather than one of minimization, the "pure" D-W sends only one vector (Pjo) to the master on each cycle k; that vector is chosen which has the highest DJ (simplex criterion). Our computational experience has shown that it is simpler to append all n vectors to the master on each cycle k rather than calculate the DJ for each. Such vectors which do not become basic may easily be dropped from the problem on subsequent cycles if computer storage limitations so require. Thus, we prefer to use the term "cycle" rather than "iteration" to describe a single execution of the four steps of algorithm.

2. Price-Endogenous planning models

Most of the models which have been solved by decomposition techniques have been, under economic interpretation, of the quota-oriented, cost-minimization variety in which the submodels have been producing units, whether sectors, oil fields, manufacturing plants, or what. As such, each of the submodels face roughly the same task - revealing its comparative advantage in the use of the scarce common resources. Thus, structurally, none of the submodels is more important than any other, and none deserve "special treatment" in a decomposition scheme.

However, when a model having a structure such as CHAC is decomposed as in Figure 3, it is apparent that the submodel of demand activities can be of particular importance if the supply responses of the producing submodels are functions of the full model's endogenous prices. In a decomposed system, the closer the price signals (i.e., those prices calculated by the master to be sent to the submodels) are to the "true" prices, the more efficient is the algorithms likely to be.

To demonstrate this heuristically, consider the following interpretation of a decomposed CHAC-type model: two primary tasks of the optimizing algorithm (whether it be standard simplex or decomposition) are to find the prices ($\bar{P}$ in Figure 2) and determine the spatial distribution of production -- the comparative advantages of the submodels. On any given cycle, the producing submodels (districts in CHAC) are price-takers in the sense that the entries of their objective functions can be considered as measures of net profit under the assumption of constant market prices. These profit coefficients depend in part on the shadow prices from the master (the π in D-W step 4; see P. 9): the π of the commodity balances are the master's evaluation of an extra unit of the output in question -- the price. Abstracting from the linear programming approximations, these prices of the outputs could be determined precisely since the exogenously given demand functions are a function of quantity only -- the quantity "supplied" by the previous production proposals (the P_{jk} vectors in D-W notation) restrained by the common constraints, if any. But within the master, the approximations to the output prices depend on the presence of the appropriate extreme points of the subprogram of demand activities. And, since true prices can be expected to fluctuate wildly in early cycles, the absence of all of the extreme points of the demand subprogram could result in unacceptably slow convergence because the prices emanating from early solutions of the master may be far from the true prices, resulting in relatively "futile" supply response of the producing submodels.

With this background, we can anticipate the major conclusion of our experiments: if a price-endogenous linear programming model is to be decomposed using the Dantzig-Wolfe algorithm, it is likely to be far more efficient (i.e., require fewer cycles to convergence and less computational time) to incorporate the subprogram of demand activities <u>directly</u> into the master program. If this is done, the master will possess all of the extreme points necessary to produce prices as close to the actual as the full model, and thus send "true" (up to the linear approximations of the full model) prices to the price-taking submodels.

There is, of course, a 'cost' associated with this procedure, and that is that the master problem will have more activities and in most applications, more constraints than in the strict D-W version. It remains to be seen, then, whether the assumed reduction in the number of master and subprogram solutions is sufficient to offset the additional computational cost of the larger master program over the course of convergence. With this in mind, we now present an initial set of experiments based on a small, somewhat artificial experimental model.

3. POQUITA: Dantzig-Wolfe applied to an experimental model

Purely out of a desire to gain experience with the computational aspects of decomposition algorithms*, a small model possessing the salient characteristics of CHAC was constructed. Although POQUITA was "distilled" from initial versions of CHAC components, no attempt will be made to justify its numerical content or its results because of its simplistic nature. Instead, we prefer to present it as a purely hypothetical example of a price-endogenous linear programming planning model.

The structure of POQUITA follows closely that of CHAC; having the same decomposable structure of producing submodels, the same objective function of maximizing the sum of producer and consumer surplus, and a set of "demand" activities which permit output prices to be endogenously determined.

The non-decomposed model has seventy-two activities and sixty constraints, three producing submodels, and nine outputs (only eight of these

* Several algorithms distinctly different from D-W have been devised in the course of the project and tested on Poquita. To date, none have shown sufficient advantage to present.

have endogenous prices; the other is assumed to face a perfectly elastic export demand). The schematic structure is shown in Figure 4.

The first set of activities, titled "demand", approximate the sum of the areas under the linear demand curves. Each curve is divided into four equal segments between arbitrary quantity bounds. A selling activity is defined for each segment as selling one ton of the output for Pij the price at the segment midpoint. b_d is the right-hand-side vector of selling activity constraints, the elements of which are the segment lengths. There are 32 selling activities plus two export activities.

The remaining activities in POQUITA are production in districts X, Y, and Z. In a given district d, these production activities are defined as cultivating one hectare in commodity i by technique k, with an OBJ entry reflecting total production costs per hectare ($-c^d_{ik}$). The yields (tons/hectare) of these activities enter the commodity balance rows.

The dimensions of the various parts of the model are:

Rows

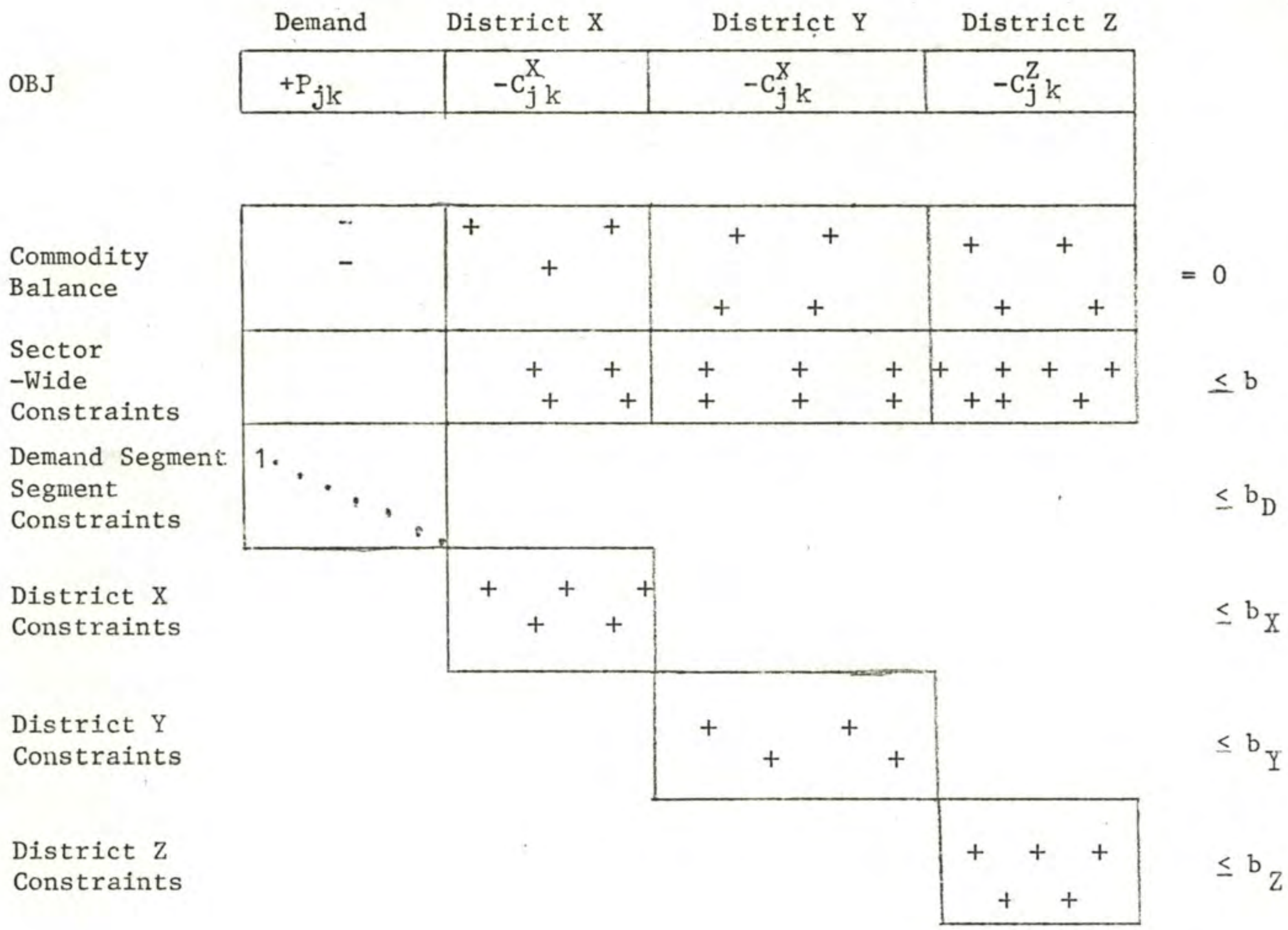
9	commodity balances
4	sector-wide resource constraints
32	demand segment constraints
5	district X resource constraints
5	" Y " "
5	" Z " "

Columns

32	demand activities
2	export activities
12	production activities, district X
14	" " " Y
12	" " " Z

Figure 4

POQUITA: Schematic Structure



3.1 Dantzig-Wolfe solution

Initially, we solved POQUITA by the straight D-W method of section III, ignoring any considerations of the economic structure which might suggest modifications to the solution procedure. Thus questions of initiating the algorithm and partitioning were arbitrary.

The three district production components were obvious choices for submodels, and the eight blocks of constraints on the eight sets of selling activities were also selected as submodels. The two export activities were incorporated into the master directly since they would otherwise have been unbounded (unless attached to a production submodel).

The algorithm was then initiated by solving each of the submodels and sending a vector from each to the master. The convergence of the maximand is shown below:

Table 1

Poquita: Dantzig-Wolfe convergence

Cycle	OBJ
1	0.00
2	274.48
3	380.62
4	423.83
5	424.34
6	428.58
7	430.84
8	431.43 (optimum)

(The Master's OBJ is 0 on the first cycle because the producing districts faced a zero vector of prices which included output prices; thus no production was profitable.)

4. Dantzig-Wolfe and the demand structure

Because of the unique role played by the demand submodels in the Dantzig-Wolfe solution (i.e., the production decisions of the districts are based on the Master's evaluation of their previous production "proposals" and can only be as good as the selling 'proposals' it possesses), we pause to consider in detail the behaviour of a demand submodel.

Consider the set of demand activities in table 2 which is based on the demand curve graphed in figure 5 .

Figure 5

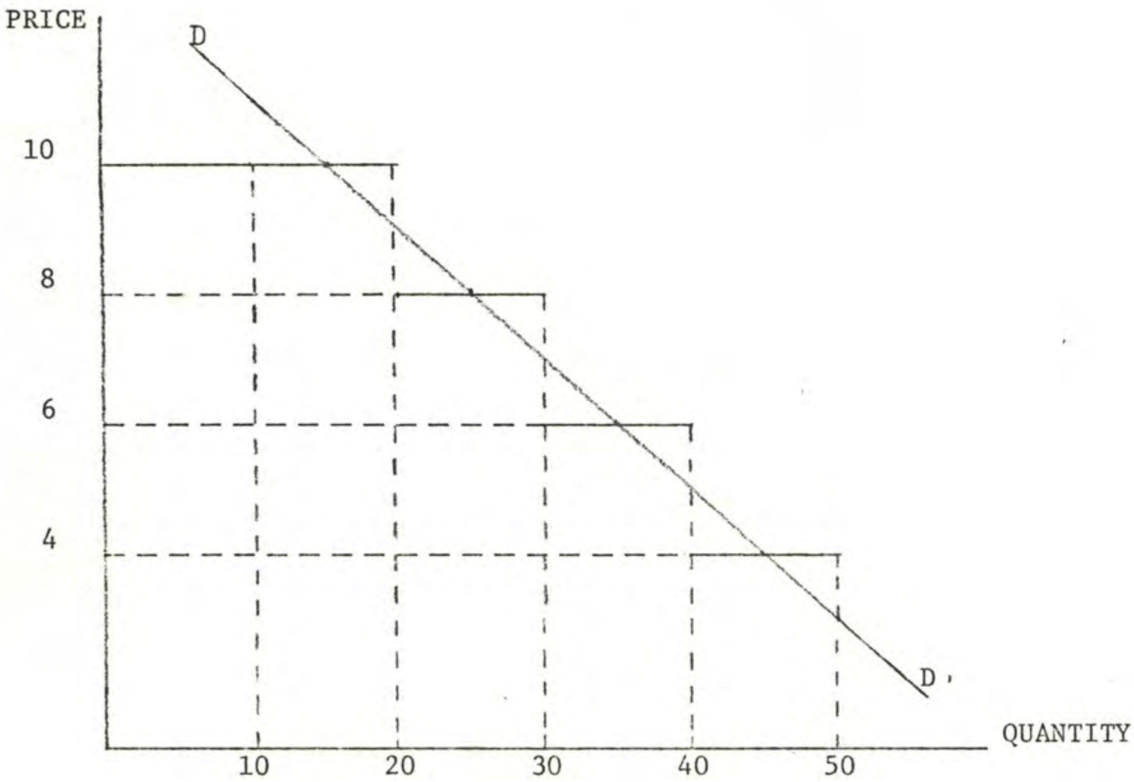


Table 2

	1	2	3	4	
OBJ	10	8	6	4	
COMMODITY BALANCE	-1	-1	-1	-1	
	1				≤ 20
SEGMENT		1			≤ 10
CONSTRAINTS			1		≤ 10
				1	≤ 10

This demand curve is divided into four segments of length 10 between quantity bounds 10 and 50 (A q less than 10 may be sold at segment one's price). If Table 2 is considered a submodel in a D-W framework, the A matrix is comprised of the commodity balance row only, and the B matrix are the segment constraints.

On the first cycle of D-W, when the π are all zero, the optimal solution of this submodel would find all selling activities used up to the bounds on their segments. The initial vector sent to the Master from this solution is shown in column one in Table 3 below.

Table 3

	1	2	3
OBJ	380	280	200
COMMODITY BALANCE	-50	-30	-20
CONVEX-COMBINATION CONSTRAINT	1	1	1

MAXIMUM π	7.6	9.33	10

Beneath this vector, in row labeled π, is the implicit price of the commodity ($380/50) in this vector. In the first solution of the Master, this is the only price of this good with which the Master has to work. Thus the shadow price on the corresponding commodity balance can be no higher than 7.6 (although it may be lower). If it is in fact 7.6, the second solution of this submodel will find activities one and two only used, resulting in the vector shown in column two of table 3 . On the subsequent Master solution, the π on this commodity's balance can be no greater than 9.33, which value would result in a solution reflected in column three of the table.

Thus if the price of this commodity in the optimal solution is greater than 9.33, it would take a minimum of three cycles just to get the appropriate extreme points of this demand submodel. If the price at optimality is, say, 6, the Master must produce a shadow price greater than 4 but less than 6 in order to obtain the necessary extreme point. The conclusion should be clear: the generation of the appropriate extreme points of demand submodels may require many cycles; in the meantime, the producing districts are making production proposals which may be based on unrealistic price signals and therefore are relatively useless. It follows that a decomposition solution should converge much faster if the Master has the full set of demand submodels extreme points at its disposal from the start; then the tasks of the algorithm reduce to finding the comparative advantages of the districts in production.

5. Lupita: Dantzig-Wolfe modified

The obvious means of giving the Master all of the demand submodel extreme points is to incorporate the set of selling activities directly into the Master, which we do in the experiments described below. For expository purposes only, we term this revision to Dantzig-Wolfe "Lupita"; although we take full advantage of the inherent flexibility of D-W, it is not a different algorithm.

The D-W Master may readily be re-defined to incorporate the selling activities and reduce the number of submodels to include only the districts. However, immediate questions concerning the size of the Master arise: the full set of demand activities may enlarge the Master (both row- and column-wise) to such an extent that solution by decomposition may not be feasible. In the case of POQUITA, which has 60 rows in the undecomposed version, the Master would have only 12 rows fewer than the full problem (15 district-specific constraints are replaced by three convex-combination constraints). It is difficult to imagine a computational situation in which decomposition would be advantageous in this case.

However, the "area" approach, suggested by Arrow and adopted for the CHAC demand structure (described elsewhere in this volume) is immediately applicable, and allows the problem to be formulated such that the Lupita Master would have no more rows than the D-W version presented above. Applying this technique to the example demand submodel of table 4 results in the tableau in table 4.

Table 4

	1	2	3	4	
OBJ	200	280	340	380	
COMMODITY BALANCE	-20	-30	-40	-50	
CONVEX-COMBINATION CONSTRAINT	1	1	1	1	≤ 1

If this submodel is incorporated into the Master, its single constraint cancels out the required D-W convex combination constraint on each submodel. Thus, if this "trick" is employed, the Lupita Master is only larger column-wise (which only affects the efficiency of a simplex solution marginally).

A review of the Dantzig-Wolfe solution procedure suggests another important modification to the basic algorithm: the initialization. The first D-W cycle is "wasted" from the point of view of the districts since no production takes places at zero output prices. If however, the Lupita Master is solved before the algorithm is initiated (i.e., when the Master is comprised of the selling activities only), the shadow prices are particularly significant: they represent the output prices in the first demand segment (the highest permissable prices) and zero prices on central resources. We term this solution* cycle zero, and initiate Lupita by sending these prices to the producing districts for their initial solutions.

The convergence of the OBJ of POQUITA decomposed using these modifications is shown below. Table 5

Poquita: Lupita Convergence

Cycle	OBJ
1	371.90
2	410.50
3	419.98
4	431.09
5	431.43

The expected improvement, cycle-wise, is evident when compared to table 1. A fuller comparison of the two solution procedures is given in table 6.

More important, in terms of real-world applications, is the rapid improvement in the early cycles. In situations where only a few cycles are possible, but good second-best solutions are required, the above modifications to D-W become highly significant. Next, we present a more realistic price-endogenous model, PACIFICO, and the results of solving it by decomposition using the Lupita modifications.

---

* This solution is trivial since all non-slack activity levels are zero. Thus the prices can be taken directly from the input data.

Table 6

(Forthcoming)

6. PACIFICO: A simulation model of Mexico's Pacific Northwest region

Out of a desire to explore the "project area vs. rest-of-world" question described elsewhere in this volume, we constructed a model of the northwest region which closely parallels the structure of CHAC. As background for discussing the decomposition experiments with this model, it is necessary to present a sketch of it.

The basic components of PACIFICO are five modified and compressed versions of the northwest distric models of CHAC, Rio Yaqui, Comision Del Fuerte, Rio Colorado, Culiacan-Humaya, and the residual district. A set of regional and export demand activities which assume that the elasticity of supply of the rest of the sector is the same as the northwest and a set of sector-wide resource supplying activities comprise the rest of the model. These sector-wide resources are supplied at fixed prices in perfectly elastic supply so that tasks of central resource allocation in a decomposition algorithm are absent, allowing us to focus on the price-determining features of a decomposition procedure.

The complete PACIFICO has 187 constraints and 382 activities, which makes it roughly equivalent in size to popular planning models.

Table 7

PACIFICO rows and columns

ROWS

4	Region-Wide Accounting Rows
16	" Commodity Balances
12	Demand Set Constraints
19	Region-Wide Resource Balances
51	Total "Central" Rows
24	Rio Yaqui - Specific Rows
26	Culmaya " "
30	Rio Colorado " "
26	El Fuerte " "
30	Residual " "
136	District-Level Rows
187	Total Rows

COLUMNS

140	Selling and "Mixing"Activities
3	Export Activities
7	Regional Factor-Supplying Activities
150	Region-Wide Activities
34	Rio Yaqui Production and Factor-Supplying Activities
45	Culmaya " " " " "
47	Rio Colorado " " " " "
39	El Fuerte " " " " "
67	Residual " " " " "
232	District-Level Activities
382	Total Activities

In the next two sections, we will examine in some detail one of the district models and the demand structure.

6.1 Production: A sample district submodel

All production activities in PACIFICO are defined on one of the five district models. The basic cropping activities were modified from those in the corresponding CHAC district models, the major changes involving the elimination in peripheral detail and a different treatment of labor hire.

Figure 6 is a picture of one of these submodels, Rio Yaqui, which we will describe for the reader who is interested in the details of generating production vectors for the Master.

The first twenty columns of the matrix are the Yaqui cropping activities which use the same notation as their CHAC counterparts. The crop acronyms comprise the third, fourth, and fifth characters of the activities. The twelve activities hire day labor on a monthly basis at a differential cost of sixteen pesos per day; FMØNCC charges all labor at ten pesos per day. Thus the farmer reservation wage is ten pesos per day, and the day laborer wage rate is 26 pesos. FGAGC charges the cost of irrigation water.

The row WELFAR is the objective function, the sum of regional consumer and producer surpluses. The remainder of the rows above the dashed line are region-wide constraints and balances; some of those not related to the Yaqui submodel are not included.

Rows S crop, where "crop" is the three-letter output acronym, are the commodity balances the entries of which are yields (tons/hectare) of the cropping activities. The next five rows are used for counting and charging the region-wide inputs credit, fertilizers and insecticides, seeds, mules, and tractor services. The commodity balances and region-wide resource balances have direct counterparts in CHAC.

Rows RDLBt are the regional day labor balances, and serve to constrain the monthly use of day labor to be less than the supply available at the minimum wage rate.* RCDL is used in charging day labor wages to the objective function and income accounting rows.

---

*Other versions of PACIFICO incorporate upward-sloping supply curves for day labor.

Figure 6

Rio Yaqui Submodel

(forthcoming)

The rows below the dashed line are Yaqui-specific. FMØNCA is the sum of the monthly labor requirements FMØNCt, and is used to charge all labor to the objective function. The FMØNCt rows are constrained by the number of farmers, but can be released through the day labor hiring activities. FGAGA is used to price the annual requirement of irrigation water; FGAGB constrains the total water use to be less than the available supply.* FTRRBt are monthly land constraints. Some of these were omitted if they were redundant.

6.2 Outputs and the demand structure

PACIFICO encompasses the production and demand of fifteen crops (one of these, cotton, is a joint product of seed and fiber, resulting in sixteen outputs in total); the prices of all are determined endogenously. Table 8 is a list of these outputs, their Spanish acronyms, and their base year prices. The optimal prices from the solution will, in general, not be the same as these base prices due to imperfections both in the model and the real world, but they are expected to be close.

In PACIFICO, the first four outputs are sold individually using the same structure as CHAC (see Section , Chapter), as were all of the outputs in POQUITA. Three of these outputs, ALG, AZU, and JIT also have export activities which permit an unlimited quantity to be sold at Northwest average CIF prices. The remaining outputs in the table are sold as members of one of the four groups, again treated in the same

---

*Because of the ability to transfer water forward intertemporally at only a small loss rate, the monthly water constraints of CHAC were excluded.

Table 8

PACIFICO Outputs

Group	Output	Model Acronym	Base Price*
(Demanded Singly)	cotton fiber	ALG	4816
	sugar cane	AZU	68
	green chili	CHV	1413
	tomatoes	JIT	1150
Grains	maize	MAI	861
	wheat	TRI	800
Forages	green alfalfa	ALV	126
	dry alfalfa	ALA	354
	barley	CEG	1014
	maize	MAI	861
	sorghum	SOR	633
Oil Seeds	cotton seed	SAL	831
	safflower	CAR	1544
	sesame	JON	2407
	soy beans	SOY	1600
"Fecolas"	rice	ARO	1134
	beans	FRI	1834

manner as in CHAC.** In order to obtain a high degree of price accuracy, the demand functions were divided into fifteen segments between 1.5 and .67 of the base price.

Recall that in the CHAC demand structure, substitution among members of a particular group was permitted between arbitrary limits while maintaining constancy of relative prices within the group. In all cases, the mixing activities resulting from the extreme bundles

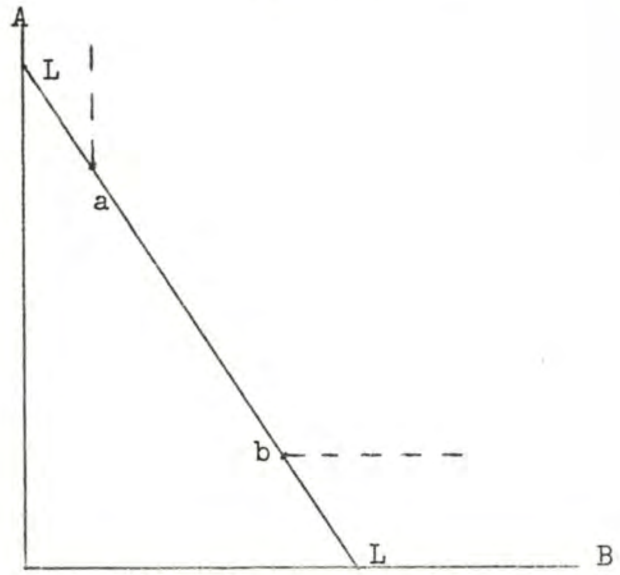
---

*1968 pesos/ton

**In PACIFICO we employed a modification to the originally presented demand structure which substantially reduced the number of activities required for the group cases and ensured that relative prices remain constant in all cases. But since the two versions are otherwise equivalents, we need not describe this modification here for our purposes.

require some positive proportion of each commodity in the group. This can be seen in Figure 7 from the familiar indifference curve analysis:

Figure 7



line segment ab represents all permissible "mixes" of commodities A and B where points a and b represent the two extreme bundles. Line ab is the indifference "curve" and is coincidental to the price line LL.

In the framework of Lupita, this structure has an important ramification: unless all of the commodities within a particular group have been produced on some previous cycle, none of the selling activities for the group can be used at a positive level. Furthermore, a production proposal which represents the production of one output in a group which is not complete (i.e., a group in which not all outputs have been represented by some production proposal) cannot be used at all. Thus the value of the Master's objective function on early cycles may not be representative of the value of the production proposals up to that point.

(For example, suppose that, on the first cycle, a given district produces mostly tomatoes, a crop which may be exported, and some alfalfa. If all of the commodities in the forages group have not been produced, then this production proposal cannot be used by the Master at all since the Master's commodity balances require strict equality of demand and supply for each output.)

If we are only employing the decomposition procedure as a means of obtaining an optimal solution to the full model, this characteristic is of no consequence. However, later we shall wish to make some pre-optimality interpretations of the Master solutions as well as comparisons of early-cycle performance of some variations to the basic algorithm, and therefore desire a means of eliminating this "curiosity."

A simple technique for accomplishing this is to add a set of "dumping" activities to the Master which permits any commodity to be disposed of at a zero price. One such activity is required for every output which is sold only in some group (there are twelve such outputs in PACIFICO). In the example of the tomatoes-alfalfa proposal above, the dumping activity for alfalfa would permit this crop to be dumped and the tomatoes sold on the domestic or export market. The Master's evaluation of a marginal unit of alfalfa would of course be zero* and this price would automatically be given to the districts on the next cycle.

* Without the dumping activity, the price of Alfalfa could be negative.

6.3 Prices and production response

Before presenting a solution to the decomposed PACIFICO, it is desirable to describe in more detail the significance of the Master's prices and the generation of district production proposals.

Recall that the objective functions of CHAC and PACIFICO yield competitive solutions in the sense that price equals marginal cost. For any solution, the output prices can be determined ex post from the primal by noting which demand segments are employed.* However, Lupita is driven by the shadow prices on the commodity balances, not the prices from the primal (although the algorithm could easily be revised to do this). Considering outputs which are demanded singly, discontinuities in both the segmented demand functions and the implicit supply functions can cause small discrepancies between the primal and dual prices. This, however, provides no problem for the algorithm since the D-W proofs of convergence are unaffected (but no proof that the algorithm converges if primal prices are used exists).

In the case of demand groups, defined such that constant relative prices must prevail and only limited combinations of the component outputs are permissible (see Figure 7), only by chance will price equal marginal cost for each output in the group. (Although this equality must hold for the group as a whole). The primal and dual prices may diverge apart from discontinuities and the prices sent to the districts may not reflect the ex ante assumptions on the demand structure concerning constant relative prices. Again, this is of no consequence for the convergence proofs, and is only relevant to an economic interpretation of the algorithm. By the usual interpretation of the dual solution, the

* Part of the output from the CHAC-PACIFICO demand submatrix generator is a table relating prices and demand segments.

commodity balance shadow prices reflect the model's marginal evaluation of a "free" unit of output, and we will continue to term this the price. Next, we briefly describe the generation of a district production proposal from a set of these prices.

Table 9 shows a sample set of output prices from a master solution.* According to the D-W rules of section 1.2, objective function coefficients for the Yaqui submodel were generated from these prices (together with the regional resource prices (not shown). These coefficients also appear in Table and reflect net profit per hectare (in model units) excluding the cost of district-specific resources land, water, and labor.

The production proposal derived from the solution with this objective function is also shown in Table 9, and is again constructed from the D-W rules. The entries in the commodity balances represent total production of the district and will work directly against the entries in the Master's selling activities. Likewise, the entries in the central rows reflect total use of these resources. Row CCCF is the Master's convex-combination constraint on the Yaqui submodel.

7. PACIFICO decomposition solution

In decomposing PACIFICO, we adopted the Lupita modifications to Dantzig-Wolfe described in Section 5. Each of the district models were of course treated as submodels, and all of the demand, export, and regional factor supplying activities were incorporated into the Master at the outset. The dimensions of the Master at initialization were 56 rows (including the five convex-combination constraints on the district subprograms) and 150

---

* These prices are actually from cycle nine of the solution described below.

Table 9

Sample Prices, District Solution, and Production Proposal

Row (Output)	Price *
SALV	105
SALG	4816
SSAL	732
SARO	1287
SALA	547
SAZU	68
SCAR	1674
SCEG	927
SCHV	942
SFRI	1843
SJIT	1200
SJON	2335
SMAI	889
SSOR	615
SSOY	1414
STRI	736

Rio Yaqui Primal Solution

Activity	Objective Function **	Use Level
FBALGA	.2900	12.8420
FBALGB	.2900	5.4653
FBALGE	.2933	0.0000
FBALGF	.2933	0.0000
FBCARA	.1263	75.2452
FBCARE	.1499	0.0000
FBJONA	.0464	0.0000
FBJONE	.0596	0.0000
FBMAIA	.1991	0.0000
FBMAIE	.2109	0.0000
FBSORA	.1610	33.2744
FBSORE	.1727	0.0000
FBSOYA	.1709	0.0000
FBSOYB	.1709	10.7132
FBSOYE	.1759	0.0000
FBSOYF	.1759	0.0000
FBTRIA	.1279	0.0000
FBTRIB	.1279	68.4048
FBTRIE	.1456	0.0000
FBTRIF	.1456	0.0000
FDL1	-.0160	0.0000
FDL2	-.0160	1.2842
FDL3	-.0160	7.0115
FDL4	-.0160	21.3483
FDL5	-.0160	0.0000
FDL6	-.0160	10.2710
FDL7	-.0160	10.2093
FDL8	-.0160	14.9349
FDL9	-.0160	0.0000
FDLO	-.0160	0.0000
FDLN	-.0160	5.1949
FDLD	-.0160	0.0000
FMONCC	-.0100	415.0141
FGAGC	-.0110	189.1700

Production Proposal

Row	Entry
WELFAR	-6.231 **
PROFIT	-2.080 **
INCOME	-2.080 **
EMPLOY	415.014
SALG	14.389
SSAL	26.014
SCAR	123.853
SSOR	164.941
SSOY	21.201
SIRI	264.179
SCRED	207.775
SICHFM	125.713
SEM	22.902
SMAQ	75.771
RDLB2	1.284
RDLB3	7.011
RDLB4	21.348
RDLB6	10.271
RDLB7	10.209
RDLB8	14.934
RDLBN	5.194
RCDL	70.254
CCCF	1.0

* In Pesos/Ton

** Profit per hectare in moble units of 10,000 pesos activities. Recall that the dimensions of the full model are 187 rows, 382 activities. Had the set of demand activities been considered as one or more subprograms, the number of rows in the master would have been reduced by at most eleven.

As with POQUITA, the algorithm is initiated by giving the district models the highest output prices (1.5 times the base price) shown in row 0 in Table 10. Also, all of the district production proposals were retained on each cycle.

Table 10 shows that the algorithm required 18 cycles to attain the optimum solution of 839.28 for the objective function, WELFAR. The behavior of the objective function over the course of the 18 cycles is interesting on two accounts: in terms of the optimum, the value of WELFAR on the first cycle (549.49) appears extremely good, and the very slow improvement after the 7th cycle.

The impressive performance on the first cycles was due largely to the presence of the entire demand set, and to the fact that the high initial prices made it profitable to produce at least something of most crops. Thus, most of the group selling activities could be used (recall that these activities required positive amounts of each component of the bundle). It is also significant that after only seven cycles, WELFAR was only about 2 1/2% from its absolute optimum, and all of the output prices were reasonably close to both the base prices and the optimum (model) prices. After the 7th cycle, progress of the objective function was extremely slow as only marginal changes in prices led to slight shifts in the district solutions. This behavior bears out the suspicion that in many cases, the degree of comparative advantage among crops and districts is very small in the northwest. Had the comparative advantages been more pronounced, they would have been exhibited much earlier, resulting

Table 10

PACIFICO Decomposition solution;

Convergence of output prices

CYCLE	WELFAR	ALG	AZU	CHV	JIT	TRI	MAI	ALV	ALA	CEG	SOR	SAL	CAR	JON	SOY	ARO	FRI
0	0.00	7224	102	2080	1725	1178	1268	186	520	1493	932	1223	2273	3544	2356	1670	2700
1	549.49	12976	221	942	2875	533	574	0	57	0	3533	0	137355	0	0	2552	4126
2	549.49	"	79	"	"	"	"	"	8264	"	267	"	944	"	22893	0	1019622
3	554.32	"	81	"	"	"	"	15864	636	"	"	"	"	247175	0	3516	1255
4	635.57	6568	"	"	1200	"	"	77	1579	58954	"	3603	901	2800	"	3530	1214
5	749.54	4816	98	"	"	513	4172	73	842	577	1343	1253	866	4566	1788	1204	1180
6	757.39	"	221	"	"	526	689	75	631	593	299	1250	889	4562	1786	1216	540
7	817.11	"	68	"	"	767	1008	156	330	1047	495	828	1746	2613	1483	1341	548
8	824.69	"	"	1295	"	736	889	105	547	927	874	732	1674	2335	1414	1287	2273
9	832.39	"	"	942	"	"	"	"	"	"	615	"	"	"	"	"	1843
10	835.05	"	"	1138	"	797	964	116	588	1041	668	804	1624	2924	1552	1401	1697
11	837.43	"	"	"	"	740	958	114	567	988	663	798	1720	2436	1540	1376	1591
12	838.32	"	"	"	"	768	"	110	573	1032	605	"	1739	2375	1541	1371	1619
13	838.55	"	"	"	"	782	989	111	"	1049	662	792	1794	2591	1531	1326	1648
14	839.04	"	"	"	"	749	923	"	555	997	631	762	1698	2600	1472	1340	1594
15	839.11	"	"	"	"	751	934	107	"	1001	643	759	1710	2582	1467	1359	1599
16	839.13	"	"	"	"	743	928	105	551	988	630	752	1669	2600	1454	1337	1586
17	839.25	"	"	"	"	742	916	109	"	986	"	753	"	2527	"	1335	1583
18*	839.28	4816	68	942	1200	756	928	110	559	1009	637	767	1706	2547	1482	1342	1606

*Optimum

in faster convergence.

Even so, one could argue that, given the eternal suspicion surrounding the numbers in such models, the solution of cycle seven is not statistically significantly better than the optimum. Of course, had we not carried the iterations through to optimality, we would not have known, say, that cycle seven was within 2 1/2% of optimality, objective function-wise. In such cases where an absolute optimum is not possible or desirable to obtain (due to computional reasons), intermediate solutions can be judged on a priori expectations as to output prices, production patterns, etc.

The reader is invited to examine the convergence of prices in Table 10. Some of the extremely high prices in the early cycles (the price of beans on cycle two was over one million pesos per ton!) was due to the treatment of substition in demand: the substitute for beans (rice) was available in abundant supply from the previous production proposals and was being "dumped"; thus production of beans, which would permit the rice to be sold, was extremely profitable.

Finally, we offer a note on the initialization of the algorithm. Intuitively, one would think that initializing the algorithm with the base year prices (the best a priori estimates of the optimum prices), would provide a better start than the "automatic" highest prices from the cycle 0 solution.

We did attempt this, with the surprising result that the objective function on the first cycle was only 189.17 as opposed to 549.49. Only seven of the fifteen crops were produced using the base prices, and none of the groups were complete.

7.1 An extension: "crashing"

With the incorporation of the Lupita modifications to the D-W algorithm, the tasks of the solution procedure reduce to finding the appropriate sets of prices to draw the required extreme points from the production models. If the appropriate price vectors were known a priori, obtaining the optimal solution would be a relatively trivial matter (ignoring the problems of discontinuities discussed above). However, we of course do not know the optimum production patterns of each of the districts before hand, nor the optimum prices. In this section, we will explore means of easily obtaining as many extreme points of the production submodels as is feasible before initiating the iterations of a decomposition algorithm.

First, let us digress to examine the computational techniques used to make the decomposition experiments on PACIFICO, in order to point out those aspects which could have a bearing on further extensions.

In the PACIFICO decomposition solution described above, the district models were solved with a FORTRAN Simplex subroutine. Such routines, though exhibiting great flexibility and ease of use to the experienced programmer, are typically limited by the size and complexity of the model. We found that, by using an advanced basis, additional district solutions could be obtained in about one second on a high-speed, third-generation computer when marginal changes to the objective function and/or right-hand-side vector were made. However, the subroutine at our disposal was not reliable in solving the PACIFICO Master, and a "canned" routine had to be used. Since such packages are generally not callable (i.e., they can only be easily used once in the course of a single run). Thus the solution procedure had to be manually interrupted after each cycle. Of course, this proves to be quite time-consuming if the required number of cycles is large. In addition, the use of a canned linear programming package entails a high fixed cost in computer time, such that the trade-off between number of individual district solutions obtained in the manner described and complete cycles, computer-time-wise, was about 30-1.

Thus it may be efficient, computationally, to generate a series of district model solutions and include these in the Master on the first cycle. It remains to decide how to select those sets of output prices for use in constructing the objective functions of the district models.

Since the demand functions in PACIFICO are linear and segmented between prices of 1.5 and .67 of the base prices (and the optimum price is expected to lie between these limits), the extreme prices of any given commodity are presumably at hand. However, since there are sixteen outputs in all, there are 2^{16} combinations of extreme prices - far too many for a realistic experiment involving district solution for each set of prices. But an examination of the demand structure can suggest ways of reducing this number to a manageable size: the three exported crops, AZU, ALG and JIT can be expected to have optimum prices equal to their export prices; thus, these could be fixed initially. Furthermore, since the technique for handling substitution in demand requires constancy of relative prices within a group of substitutable commodities, it seems logical to change the prices of commodities within a group together. If this is done, the number of combinations of extreme prices is reduced to 2^5 (exported crops fixed at export prices, one single commodity and four groups).

Thus we conducted an experiment whereby we generated 32 solutions of each district model and appended the 160 vectors constructed therefrom to the Master. The solution to this problem had an objective function value of 684.82, a value not attained until cycle five by Lupita. However, two demand groups could not be used because none of the 160 proposals included the production of sesame or dry alfalfa.*

We then generated a set of proposals for each district based on the prices from this master solution in the same manner as an ordinary Lupita cycle. The two needed crops were brought in, and the Master's objective function climbed to 822.77 - less than 2% of the overall optimum.

These results would suggest that, if the number of crucial outputs (or inputs, if the prices of these are also endogenous) is small, substantially faster convergence can be obtained by generating a series of submodel solutions based on combinations of extreme prices.

---

* The shadow prices on the commodity balances of these crops were extremely high, as expected.

7.3 Conclusions to the numeric experiments

The general conclusion that can be drawn from the decomposition experiments above is simple: if a linear programming model is to be solved by decomposition, substantial savings may be obtained by examining the economic structure of the problem for relationships which could influence the computational procedure.

Up to this point, at least, we are safely within the bounds of Kornai's "a-type" utilization of decomposition (the computational), but with an important difference: we have employed our understanding of the economic structure of the model to improve the performance of the mathematical-computational procedure. This difference is perhaps the reverse of Kornai's "b-type" utilization in that instead of interpreting a decomposition technique as an iterative economic planning technique, we have used our knowledge of the economy as implied by the model to suggest variations to the mathematical algorithm.

In particular, we have demonstrated that if a price-endogenous model is to be decomposed, substantial computational savings may be obtained if the price-determining submatrix is incorporated directly into the Master. Furthermore, the number of cycles required to produce a meaningful solution can be reduced by generating several production proposals from crucial extreme prices before initiating the formal algorithm.

Nothing, however, has been said concerning the decision to decompose or not-to-decompose. Our experience* allows us to make a hearty agreement with Kornai: if a model can be solved in its complete form with available

---

*The author also programmed the algorithmic experiments.

computational facilities, then decomposition is not likely to be preferable on computational grounds. Decomposition, when "tailored" to the structure involved, can be an effective means of overcoming computational inadequacies when, say, many solutions of a simulation or planning model are desired.*

We have not attempted to evaluate the ability of a decomposition algorithm to aid in the understanding of planning processes; Professor Kornai has covered this subject to our satisfaction.

---

*The collection of previously generated production proposals can serve as an advanced starting point for additional solutions involving parametric changes to output and input prices and central resource supplies.

INTERNATIONAL DEVELOPMENT ASSOCIATION | INTERNATIONAL BANK FOR RECONSTRUCTION AND DEVELOPMENT | INTERNATIONAL FINANCE CORPORATION

OFFICE MEMORANDUM

DRC file

TO: Mr. Pedro-Pablo Kuczynski

DATE: September 16, 1971

FROM: L. M. Goreux

SUBJECT: Dinamico - An Economy-Wide Programming Model of the Mexican Economy

The attached note summarizes some of the results obtained from Dinamico. As you will recall, Y. Franchet and H. Bergendorff went to Stanford for a week to work on Dinamico with A. Manne. The objective was to reduce the communication gap between the Development Research Center and the Mexican economic mission. You will note that in the last section, an attempt is made to analyze the trade-off between growth and income distribution. I would appreciate receiving your reactions.

You will receive under separate cover A. Manne's memo 71-12, 71-13 and 71-14.

cc: Messrs. Carter
Hayes
Lerdau
Pfeffermann

SOME EXPERIMENTS WITH DINAMICO,
AN ECONOMY-WIDE MODEL OF THE MEXICAN ECONOMY

The object of this note is only to provide a flavor of some of the results 1/obtained from DINAMICO, a fairly complex model developed by Professor A. Manne for the Mexican economy. The first section compares, for the period 1971-1980, the macro-economic results obtained from DINAMICO and from the Economic Program Department simulation model used in the last IBRD economic report on Mexico. The second section describes for 1980 some of the informations which are generated by DINAMICO but not by the E.P.D. simulation model. Those refer to the sectorial production breakdown, the labor skill composition, the shadow prices and the implications regarding income distribution. The first and second sections refer only to the basic solution of DINAMICO; the third section, which is based on the comparison between the basic solution and thirteen alternatives, analyzes various types of trade-offs.

1. Comparison between Dinamico and the Simulation Model

Dinamico is a linear programming model maximizing consumption. It consists of six three-year periods starting with 1968-1971 and ending 2/ with 1983-1986. It includes fifteen sectors; it draws a distinction between five labor skill categories but not between private and public sectors.

The E.P.D. simulation model focuses on the problems of public expenditures, balance of payments and debt servicing. It is solved recursively year by year and covers the period 1971-1985. GDP and export growth, which are endogenous in Dinamico, are exogenous in the basic case 3/ of the EPD simulation.

Table 1 provides a comparison between the basic solutions of the two models for the period 1971-1980, which was selected as the most relevant. Despite the differences in the conceptions of the two models, the results are very similar. The two main differences are: (a) the higher saving propensity in Dinamico which reflects a higher capital-output ratios and; (b) the greater trade surplus in Dinamico on account of higher exports (+4.9) due to larger net factor payments (+4.6).

2. Basic Solution for 1980

2.1 Sectoral Breakdown

The absolute values of production and investment in 1971 and 1980 and the average growth rates are shown on Table 2 for 15 sectors. Agriculture is the slow growing sector (+4.1% a year). Chemicals, basic metals, machinery and electricity are the fast growing sectors (10 to 11% a year).

1/ The basic documents, A. Manne Memoranda No. 71-9, 71-12, 71-13 and 71-14, and the computer printout are available on request in the Development Research Center.

2/ Terminal conditions are specified for the period 1986-1989.

3/ Alternative cases to the EDP simulation model are used to investigate the impact on debt servicing of variations in the growth of GDP and exports and in fiscal measures.

2.2 Labor Skill Composition

As appears from the last two columns of Table 3, the number of unskilled agricultural workers (skill 5), which increased by 1.8 percent a year between 1960 and 1968, remains stable between 1971 and 1980. The growth rate is slightly smaller between 1971 and 1980 than between 1960 and 1968 for the highest two skills (1 and 2), while the reverse is true for skills 3 and 4.

2.3 Shadow Prices and Income Distribution

As shown in Table 4, the shadow wages generated by the model over the period 1971-1983 are generally higher than the wages estimated for 1960, especially for the highest two skills and for agricultural workers. As shown in Table 7, the increase in total labor income between 1960 and 1980 can be decomposed in three elements: (a) the size of the labor force, (b) the average wage per skill, using the 1960 skill distribution as weight and, (c) the modification in the skill composition. As appears from the first column of Table 7, the modification in the skill composition has more impact (1.7% a year) on the wage bill than the increase in average wages (1.3% a year). Total labor income rises less rapidly than total consumption: its share falls from 62 percent in 1960 to 56 percent in 1980.

The premium on foreign capital equals 15 percent throughout the period 1971-1980, while the shadow interest rate on foreign capital 1/ varies from 20 to 16 percent.

3. Sensitivity Analysis

Modifications to the assumptions made in the basic case have been introduced one at a time. They lead to thirteen alternative solutions 2/. Although some of the modifications introduced are rather extreme, they do not have a large impact on the growth of the Mexican economy. Thus, the yearly rate of GDP growth always remains within the narrow 6.7 to 7.1 percent range 3/. This finding is typical of large-scale models; relaxing one constraint allows to increase growth only as long as another constraint does not become binding. While overall growth is not very sensitive to the various alternatives, some particular variables are. This leads to the analysis of a number of trade-offs described below.

1/ Own rate of interest on the shadow prices of foreign exchange for two different years.

2/ Memorandum 71-14 contains alternatives 1 through 8. Alternatives 9 through 13 are available only on working sheets and computer printout.

3/ Except for case (7) presented in Memorandum 71-14 for didactic purposes.

3.1 Nature of the Objective Function, the Trade-off between Present and Future Consumption

Cases (0) through (4) shown in Table 5 correspond to five alternative formulations of the objective function, namely: maximizing consumption subject to a gradual consumption path with asymptotic growth rates of 6, 7 (basic case) and 8 percent, maximizing the discounted consumption flow, maximizing terminal consumption. The first three lines of Table 5 show that the variations in the level of consumption among these five cases remain small in 1971, 74 and 80. This lack of sensitivity is due to the inclusion in the model of a 30-percent upper bound on the marginal propensity to save. This constraint, which is never binding in case (0), becomes critical in cases (3) and (4).

Case (1) may be interpreted as the soft option providing more in terms of consumption goods and requiring less in terms of fiscal efforts during the first decade. By contrast, cases (2), (3) and (4) are hard options. The trade-off between harder fiscal measures today and more consumption tomorrow appears from the last four lines of Table 5.

3.2 Export Subsidy for Supplementary Manufacturing Exports, the Trade-off between Manufacturing and Agricultural Exports

In all cases but (5), it is assumed that supplementary manufacturing exports can be promoted with an initial export subsidy of 30 percent falling progressively to zero within 18 years. The value of the premium on foreign capital is very sensitive to this assumption. Thus, when it is assumed in case (5) that the initial export subsidy required is 50 instead of 30 percent, the premium in 1980 rises from 15 to 35 percent. The result is a reduction of manufacturing exports which are replaced by agricultural exports. The impact on GDP and on most other primal and dual variables is insignificant.

3.3 Shadow Interest Rate and Terms of Commercial Borrowing from Abroad

In Dinamico concessional borrowing is exogenous, while non-concessional borrowing is endogenous. The impact of the terms of non-concessional borrowing appears from the comparison of cases (0), (6) and (11) on Table 6.

In cases (0) and (6), non-concessional borrowing costs 15 percent per year in terms of interest charges and profit remittances. During each of the first and last time periods, non-concessional borrowing is fixed at 5 billion pesos. During the four intermediate periods, the amount borrowed is endogenous but cumulated borrowing cannot exceed 20 billion pesos.

In cases (0) and (6) the shadow interest rate is above 15 percent; therefore, the model uses the maximum amount of 20 billion pesos. In case (6) when the model is free to choose the timing at which these 20 billion are borrowed; Mexico borrows the quasi-totality in the first two intermediate periods. This bunching in borrowing leads to a bunching in the creation of new export outlets and to an irregular time path of the premium on foreign capital. In case (0), Mexico cannot borrow more than 5.5 billion pesos in any of the intermediate periods. It borrows the maximum in the first three and the balance in the fourth one. Imposing a regular time path for borrowing leads to a regular time path for the premium on foreign capital and for the shadow interest rate, at the cost of a very small loss in consumption.

In case (11), the cost of non-concessional borrowing is reduced from 15 to 10 percent; this might be interpreted as a correction for inflation. Cumulated non-concessional borrowing is bounded to keep comparable terminal debt conditions. Borrowing remains fixed in the first and last periods, but is free in the four intermediate periods. To avoid bunching, bounds are placed on the capacity to expand supplementary export in each period. The premium on foreign capital disappears in 1974 and 1971 but rises steeply in 1980 and 1983 when the loans have to be reimbursed. The total amount of non-concessional borrowing is at its maximum, which is 17 billion pesos higher in case (11) than in case (0). As a result, consumption in 1986 is 8 billion pesos higher in case (11) than in case (0).

To conclude, when Mexico can borrow at 15 percent in cases (0) and (6), the shadow interest rate is equal to 19.6 percent in case (0) and 17.4 percent in case (6); the 2.2 percent decline in case (6) reflects the relaxation of the timing constraints on borrowing. When it can borrow at 10 percent, in case (11), the shadow interest drops to 12.8 percent, which still exceeds the interest rate paid.

why?

3.4 Capital-Labor Substitution, the Trade-off between Growth and Equity

When all labor constraints are eliminated from Dinamico (case 7), the shadow price of labor falls to zero and the shadow interest rate rises to 32 percent (100/capital-output ratio). This pitfall common to most macro-economic models is avoided in Dinamico by introducing labor constraints for five skill categories.

Total labor supply is exogenous. Unskilled labor can be upgraded through endogenous educational activities. Labor requirements by skills are computed by multiplying, for each sector, the volume of production by exogenous labor norms adjusted through time to reflect improvements in labor productivity. To relax the rigidity which would have resulted from the lack of technological choices within each sector, capital-deepening activities have been introduced.

In cases (0) through (11), a capital deepening activity 1/ exists only in the agricultural sector. Workers released from agriculture can be transferred to the urban sector at a yearly fixed cost in terms of urban services. In cases (12) and (13), capital deepening activities are available both in the urban and the rural sectors. The level of rural-urban migrations, which was endogenous in cases (0) through (11), becomes exogenous in cases (12) and (13).

For a given skill, the shadow wage is equal to the marginal product of a worker in terms of consumption. The unemployment problem is treated as a low labor productivity problem. It is measured in terms of income distribution, as shown in Table 7.

The trade-off between efficiency and equity may be illustrated by the comparison between cases (0) and (8) shown in Tables 7 and 8. In case (0), all investments are selected according to efficiency only. Between 1968 and 1980, agricultural output rises by 4.1 percent a year, while the number of agricultural workers slightly declines. Due to the rapid rural-urban migration flow thus induced, the wages of unskilled urban workers remain low and the share of labor income falls from 62 percent in 1960 to 56 percent in 1980.

In case (8), the amount of capital used to replace an agricultural worker is twice as large as in case (6). This can be interpreted as a situation where public investments in agriculture are selected with a view towards improving the "quality of life" in the most deprived rural areas. A number of production oriented investments (irrigation schemes) selected in case (0) are therefore replaced by welfare oriented investments (running water in villages). Due to the food self-sufficiency constraint imposed in Dinamico, the agricultural sector must produce approximately as much in case (8) as in case (0). As a result of the different choice of public investments in case (8), agriculture needs more workers and the supply of unskilled urban workers is reduced. The result is a large increase in the wages of unskilled workers and in the share of labor-income, at the cost of a small decline in the growth of total consumption. The comparison of cases (0) and (8) shows that for a one-unit loss in total consumption, unskilled rural and urban workers gain 7.5 units. The trade-off will be less favorable in the real world because the shadow prices of Dinamico are much more sensitive than the actual wages. Nevertheless, this comparison illustrates how technological choices 2/ may affect income distribution.

1/ The amount of capital required to replace a worker through capital deepening activities increases by 2 percent a year. Within any given period, the amount of capital required to replace a worker remains unaffected by the number of workers replaced in cases (0) through (12). But in case (13), it **rises** with the number of workers replaced. Decreasing return to capital is therefore recognized only in case (13).

2/ The efficiency loss in public investments is reflected by a decline in the shadow interest rate. This may not however affect the choice of private investors, if the government discourages labor saving investments through an appropriate tax policy.

The comparison between cases (0) and (11) shows that growth and labor income shares increase simultaneously, if Mexico can borrow more foreign capital at better terms. The comparison between the first two columns of Table 7 indicates that two thirds of the labor gain is due to a wage increase and one third to skill upgrading. More growth induces a rise in unskilled wages and an acceleration of rural-urban migrations, as shown in Table 8.

The comparison between cases (0) and (13) illustrates how a reduction 1/ in rural-urban migrations coupled with a decreasing return to capital can improve labor share (strictly on account of higher wages) without affecting total growth. Clearly, income distribution is very sensitive to the assumptions made on labor mobility between the urban and the rural sectors and on the cost of capital-labor substitution within each sector. Improving our knowledge on these points is essential for a better analysis of the trade-off between growth and income distribution.

Rather than perfecting Dinamico by increasing its size and complexity, it was found preferable to build up desaggregated sectoral models which can be linked with it. Thus, highly desaggregated models have been built for energy and agriculture. The completion of the agricultural model will provide an objective estimate of the capital labor substitution curve in agriculture. This will improve considerably the assessment of the trade-off between growth and income distribution. Moreover, it will provide a much more concrete basis for analysing the impact of policies on private and public investments.

1/ In case (13), the exogenous rate of growth of the absolute number of the agricultural labor force declines steadily from 1.6% a year in 1970 to zero in 1990.

Table 1: COMPARISON BETWEEN DINAMICO a/ (A. Manne) AND THE SIMULATION b/ MODEL (E.P.D.)

		1971 Sim.	1971 Din	1980 Sim.	1980 Din.	Average Yearly Growth Rate 1971-1980 Sim.	Average Yearly Growth Rate 1971-1980 Din.
		(in billions of 1960 pesos)				(in %)	
(1)	GDP	317.0	319.7	582.6	582.6	7.0	6.9
(2)	Consumption	255.0	251.4	461.2	445.8	6.8	6.6
(3)	Investment	63.3	70.8	117.7	128.2	7.1	6.8
(4)	Saving	62.0	68.3	121.4	136.8	7.8	8.0
(5)	Imports	23.4	25.8	41.2	41.2	6.5	5.3
(6)	Exports	22.1	23.3	44.9	49.8	8.2	8.8
(7)	Resource Gap	+1.3	+2.5	-3.7	-8.6	-	-
(8)	Net Factor Payments	-6.6	-6.8	-14.0	-17.2	7.5	10.9
(9)	Non-concessional c/	+5.5	+5.0	+8.9	5.5	3.5	1.1
(10)	Concessional Capital Inflow	+2.4	+4.3	+1.5	3.1	-	-

(1) = (2) + (4) (7) = (5) - (6) = (8) + (9) + (10) = (3) - (4)

a/ Basic case b/ Basic case, soft terms. c/ not strictly comparable in the two models

Table 2: SECTORAL OUTPUT AND INVESTMENT

Sectoral	Gross Production 1971	Gross Production 1980	Gross Production Average Yearly Growth Rate	Investment 1971	Investment 1980	Investment Average Yearly Growth Rate
	(in billions of 1960 pesos)		(in %)	(in billions of 1960 pesos)		(in %)
1. Agriculture	51.946	74.615	4.1	11.47	12.17	0.7
2. Mining	5.716	8.593	4.6	.26	.43	5.7
3. Petroleum	20.362	38.964	7.5	2.66	5.62	8.7
4. Food	59.341	112.799	7.4	4.28	8.14	7.4
5. Textiles	27.266	54.759	8.0	1.86	3.76	8.1
6. Wood	11.887	22.568	7.4	1.22	2.35	7.6
7. Chemicals	23.093	54.767	10.1	3.18	7.29	9.7
8. Non-metallic	6.977	14.935	8.8	.76	1.86	10.5
9. Basic metals	14.554	37.010	10.9	2.91	7.06	10.3
10. Machinery	38.861	100.074	11.1	4.54	10.61	9.9
11. Construction	34.462	59.055	6.2	.32	.78	10.4
12. Electricity	7.672	19.818	11.1	4.29	11.64	11.7
13. Commerce	109.380	196.719	6.7	3.90	7.73	7.9
14. Transportation	14.702	24.152	5.7	8.13	13.76	6.0
15. Services	88.676	160.993	6.9	21.03	35.00	5.8

Table 3: SKILL COMPOSITION OF THE LABOR FORCE

Labor by Skill	1960	1968	1971	1980	60-68	71-80
	(thousands)				(% per yr.)	
1. Engineers and Scientists	37	77	95	183	9.6	7.6
2. Other Professional and Technical Workers	373	646	774	1,310	7.1	6.0
3. Administrative and Clerical Workers	915	1,382	1,633	2,673	5.3	5.6
4. Manual and Sales Workers outside Agriculture	3,916	5,451	6,277	9,521	4.2	4.7
5. Unskilled Agricultural Workers	6,091	7,024	6,709	6,765	1.8	.1
Total:	11,332	14,580	15,488	20,452	3.2	3.1

Table 4: SHADOW PRICES OF LABOR, FOREIGN EXCHANGE

Labor by Skill	1960	1971	1974	1977	1980	1983
	(thousands of 1960 pesos of consumption)					
1. Engineers and Scientists	72.0	79.3	78.3	110.1	106.9	119.2
2. Other Professional and Technical Workers	30.0	37.8	37.3	55.7	58.1	46.4
3. Administrative and Clerical Workers	21.6	22.6	22.4	21.2	19.3	28.5
4. Manual and Sales Workers outside Agriculture	7.2	7.6	7.5	6.4	8.4	7.9
5. Unskilled Agricultural Workers	2.4	4.1	4.0	2.7	4.7	4.3
Efficiency Price of Foreign Capital Relative to Price of New Supplementary Exports		1.15	1.15	1.15	1.15	1.24
Own rate of Interest of Foreign Capital		19%	20%	19%	16%	

Table 5: Nature of the Objective Function

	Gradual consumption path with asymptotic rate of: 6% (1)	7% (0)	8% (2)	Discounted consumption flow (3)	Terminal consumption (4)
	(------------------	billions of	1960 pesos	-------------	)
Consumption					
1971	254.1	251.4	248.7	245.3	245.9
1974	308.1	303.6	298.9	298.9	295.8
1980	448.8	445.8	441.9	442.6	434.4
1986	648.6	659.3	668.7	692.7	700.1
Savings					
1971	66.2	68.3	67.4	67.7	68.0
1974	78.3	81.9	89.0	90.7	89.4
1980	123.9	136,8	150.2	152.3	148.8
1986	172.4	193.3	216.2	206.5	199.4
GDP					
1971	320.3	319.7	316.1	313.0	313.9
1974	386.4	385.5	387.9	389.6	385.2
1980	572.7	582.6	592.1	594.9	583.2
1986	821.0	852.6	884.9	899.2	899.5
	(------------------	percent per year	------------------		)
GDP growth					
1971 - 1980	6.7	6.9	7.2	7.4	7.1
1980 - 1986	6.2	6.5	6.9	7.1	7.5
Consumption growth					
1971 - 1980	6.5	6.6	6.6	6.8	6.5
1980 - 1986	6.3	6.7	7.1	7.8	8.3
Marginal propensity to save	(---------------------	percent	---------------------		)
1971 - 1980	22.9	26.1	30.0	30.0	30.0
1980 - 1986	19.5	20.9	22.5	17.8	16.0

Table 6: IMPACT OF TIMING AND TERMS OF BORROWING

Case	Constraints	Item	Units	1971	1974	1977	1980	1983	1986
(0) Basic Case	r=15%	(a) Consumption	billion 1960 pesos	251.4	303.6	367.5	445.8	541.8	659.3
	$\sum_t FDP_t \leqslant 30.0$	(b) Non-concessional capital inflow	"	5.0	5.5	5.5	5.5	3.5	5.0
	$FDP_{71}=5.0$	(c) Resource gap	"	2.5	0	-4.0	-8.6	-15.5	-17.6
	$FDP_{80}=5.0$	(d) New Supplementary Exports	"	1.8	2.4	4.0	3.8	4.6	0
	$FDP_t \leqslant 5.5$ (t=74,77,80,83)	(e) Shadow Premium on Foreign capital*	%	15	15	15	15	24	16
		(f) Shadow interest rate on Foreign capital between 1974 and 1980**	% per yr.		-----	19.6	-----		
(6) Borrow Earlier	r=15%	(a) Consumption	billion 1960 pesos	251.4	303.7	367.7	446.0	542.1	659.7
	$\sum_t FDP^t \leqslant 30.0$	(b) Non-concessional capital inflow	"	5.0	9.9	9.3	.8	0	5.0
	$FDP_{71}=5.0$	(c) Resource gap	"	2.5	4.4	-2.5	-17.7	-20.8	-17.6
	$FDP_{86}=5.0$	(d) New Supplementary Exports	"	1.5	0	7.1	11.2	0	0
		(e) Shadow Premium on Foreign capital*	%	22	5	13	21	33	4
		(f) Shadow Interest Rate on Foreign capital between 1974 and 1980**	% per yr.		-----	17.4	-----		
(11) Borrow Cheaper	r=10%	(a) Consumption	billion 1960 pesos	252.2	305.3	370.3	450.0	547.7	667.3
	$\sum_t FDP^t \leqslant 47.2$	(b) Non-concessional cap. inflow	"	5.0	12.7	12.4	12.1	0	5.0
	$FDP_{71}=5.0$	(c) Resource gap	"	2.5	8.2	2.5	-3.6	-21.8	-18.6
	$FDP_{86}=5.0$	(d) New Supplementary Exports		.3	0	4.0	4.0	4.0	3.5
	$ZM_t \leqslant 4.0$	(e) Shadow Premium on Foreign capital*	%	33	-16	4	30	34	22
		(f) Shadow Interest Rate on Foreign capital between 1974 and 1980**	% per yr.		----	12.8	-----		

<u>Definitions:</u> FDP: non-concessional capital inflow
ZM: supplementary exports of manufactured goods
r: rate of interest and profit remittances on FDP

* $100\left(\frac{P_F}{P_{ZM}} - 1\right)$ where P_F is the shadow price of foreign exchange and P_{ZM} is the shadow price of new supplementary exports

** $P = \left(\frac{P_{F74}}{P_{F80}}\right)^{1/6} - 1$

Table 7: LABOR INCOME GROWTH* 1960-1980

	Ob. (0)	(11)	Eq. (8)	(12)	(13)
	Rural Capital Deepening Migrations Endogenous			Rural and Urban Capital Deepening. Migrations Exoge	
	Basic Case	More and Cheaper Non-concessional Capital	Double Cost of Capital Labor Substitution	Fixed Capital Returns	Diminishing Capital Returns
	(		% per year		)
a. Total Number of the Labor Force	3.0	3.0	3.0	3.0	3.0
b. Average Wage with the 1960 Skill Distribution	1.3	1.5	3.0	2.1	2.3
c. Change in Skill Distribution	1.7	1.8	1.1	1.4	1.7
d. Total Labor Income*	6.1	6.4	7.2	6.7	7.1
e. Total Consumption	6.6	6.7	6.5	6.6	6.6
f. Share of Labor Income in 1980	56%	59%	70%	62%	67%

*Using ' for 1980 and no ' for 1960, Ns for the number of workers of skill s at a shadow wage w_s, the total wage bill in 1960 is $W = \sum_s w_s N_s$. The wage bill ratio can then be decomposed in three components:

$$\frac{W'}{W} = \frac{N'}{N} \times \frac{\sum_s w'_s \frac{N_s}{N}}{\sum_s w_s \frac{N_s}{N}} \times \frac{\sum_s w'_s \frac{N'_s}{N'}}{\sum_s w'_s \frac{N_s}{N}}$$

In relation to the yearly percentages shown in the table, this ratio can be rewritten:

$$(1+ .01d)^{20} = (1+ .01a)^{20} (1+ .01b)^{20} (1+ .01c)^{20}$$

Calling C total consumption, the other two coefficients are defined by:

$$(1+ .01e)^{20} = \frac{C'}{C} \text{ and } f = 100 \frac{W'}{C'}$$

Table 8: INDICES OF EMPLOYMENT AND WAGE LEVELS IN 1980 FOR UNSKILLED WORKERS (100=Basic Case) AND SHADOW INTEREST RATE 1974-1980

		Rural Capital Deepening Migrations Endogenous			Rural and Urban Capital Deepening. Migrations Exogen	
		Eff. (0)	(11)	Equity (8)	(12)	(13)
	Reference 1968 for Employment 1960 for Wages	Basic Case	More and Cheaper Non-Concessional Capital	Double Cost of Capital Labor Substitution	Fixed Capital Returns	Diminishing Capital Returns
Unskilled Agricultural Workers						
Employment	104	100	96	107	121	117
Wage	62	100	102	202	94	82
Unskilled Non-Agricultural Workers						
Employment	57	100	108	97	85	88
Wage	96	100	105	150	167	167
Shadow Interest Rate Between 1974 and 1980 (% per year)		20%	13%	15%	18%	15%