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OPTIONS FOR PRODUCING "SMOOTHENED" PPP TIME-SERIES FOR THE YEARS BETWEEN REFERENCE YEAR COMPARISONS

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1. The Problem

Here we consider the specific problem of constructing interpolated PPP time series between two benchmark years. This problem arises in the current context of building time series between the 2011 benchmark and the current 2017 benchmark round which is expected to be completed by mid-2019. The main inputs into this process are:

- Matrices of PPPs for 155 ICP basic headings for the years 2011 and 2017. It is also possible to consider interpolation at higher levels of aggregation.
- Matrices of expenditures, in national currency units, at the basic heading level for all the participating countries. Typically information available here is in the form of national accounts weights but using estimates of GDP it is possible to identify expenditure and the basic heading level.
- Implicit price deflators from national accounts that can be derived from the data expenditure at the basic heading level at current and constant prices expressed relative to a base year.

The problem is one of constructing matrices of BH PPPs for the years in between 2012 and 2016 taking into account all the information available in the forms of inputs listed above. Apart from the technical or statistical problem interpolation, we need to be cognizant of varying quality of the input matrices.

- BH PPPs for the two benchmarks 2011 and 2017 may be considered reliable.
- Expenditures and shares at the basic heading level are of differing qualities across different participating countries. This means that one needs to balance the need to use these data in the interpolation process against the possibility of introducing serious measurement errors in this process.
- Implicit deflators are usually reliable for higher level aggregates than at the basic heading level. In many instances it may be necessary to map each basic heading to a suitable higher level aggregate so that deflator for the aggregate can be used as a proxy.

Data for this interpolation are being compiled by the ICP Global Unit at the World Bank. In what follows, we abstract from the quality and availability issue and simply focus on the options available for interpolation and construction of PPPs for the intervening years.

2. What is an optimal level for interpolation?

We anchor our discussion on the discrepancy between actual price comparisons and comparisons based on extrapolations and draw on the earlier work of Inklaar and Rao (2017).

We consider the simple case of two countries where PPP is computed using Törnqvist index numbers. For simplicity, we assume that the same set of commodities enter PPP and national level index number computation. We also assume that the expenditure shares of commodities differ across countries but remain the same over time periods t and $t+1$. Let p_{ij}^{s} represent the price of the i^{th} commodity $(i = 1,2, \dots, N)$ in country $j\ (= 1,2)$ in period $s\ (s = t$ or $t+1)$. Let s_{ij} represent expenditure shares associated with commodity i in country $j\ (j = 1,2)$.[1] We further let PPP_2^s represent purchasing power parity of currency of country 2 with country 1 as the reference country in period s.[2] Let P_j represent the price index in country j (1 and 2) over time t to $t+1$. Then the logarithmic form of the three Törnqvist indices are given by:

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[1] We do not have time superscript with expenditure share as we assume that expenditure shares remain the same over time. Expenditure shares tend to move slowly over time, so this is not a tenuous assumption.

[2] We drop subscript 1 with PPP for ease of notation.

$$\ln PPP_2^s = \frac{1}{2}\sum_{i=1}^{N}(s_{i1} + s_{i2})(\ln p_{i2}^s - \ln p_{i1}^s) \text{ for } s = t \text{ or } t+1 \tag{1}$$

$$\ln P_2 = \sum_{i=1}^{N} s_{i2}\left(\ln p_{i2}^{t+1} - \ln p_{i2}^t\right) \tag{2}$$

$$\ln P_1 = \sum_{i=1}^{N} s_{i1}\left(\ln p_{i1}^{t+1} - \ln p_{i1}^t\right) \tag{3}$$

It is easy to see that PPP_2^s is a Törnqvist index that compares price levels across countries 1 and 2 whereas P_1 and P_2 represent Törnqvist indices for countries 1 and 2 measuring price changes from t to $t+1$.

Following Deaton (2012), we consider the change in PPP over time in logarithmic form. This is given by:

$$\ln PPP_2^{t+1} - \ln PPP_2^t = \frac{1}{2}\sum_{i=1}^{N}(s_{i1} + s_{i2})\left[\left(\ln p_{i2}^{t+1} - \ln p_{i1}^{t+1}\right) - \left(\ln p_{i2}^t - \ln p_{i1}^t\right)\right] \tag{4}$$

After simple rearrangement and definitions in (1), (2) and (3), we can show that equation (4) equals:

$$\ln PPP_2^{t+1} - \ln PPP_2^t = \ln P_2 - \ln P_1 - \frac{1}{2}\sum_{i=1}^{N}(s_{i2} - s_{i1})\left[\ln\left(\frac{p_{i2}^{t+1}}{p_{i2}^t}\right) + \ln\left(\frac{p_{i1}^{t+1}}{p_{i1}^t}\right)\right] \tag{5}$$

From equation (5), inconsistency between benchmark and updates is given by:

$$\ln PPP_2^{t+1} - \ln PPP_2^t - (\ln P_2 - \ln P_1) = -\frac{1}{2}\sum_{i=1}^{N}(s_{i2} - s_{i1})\left[\ln\left(\frac{p_{i2}^{t+1}}{p_{i2}^t}\right) + \ln\left(\frac{p_{i1}^{t+1}}{p_{i1}^t}\right)\right] \tag{6}$$

Deaton (2012) argues that this inconsistency depends on the covariance between differences in expenditure shares in the two countries and price movements in prices in the two countries under consideration.

However, we consider a different angle for equation (6). If the N commodities considered here represent a commodity group, we ask the question as to when the inconsistency between updates and benchmarks is likely to zero or very small. The following result provides a useful direction.

Result 1: Under the set-up considered in equations (1) to (6) based on Törnqvist index for the measurement of price levels across countries and price change over time, inconsistency between benchmarks and updating using domestic measures of price changes vanishes if all the commodities considered in the computation show the same price change over time.

In order to verify this result, suppose prices of all the commodities in country 2 change by the same percentage α and price change is uniform across commodities in country 1 represented by a percentage change β, then equation (6) becomes:

$$\ln PPP_2^{t+1} - \ln PPP_2^t - (\ln P_2 - \ln P_1) = -\frac{1}{2}\sum_{i=1}^{N}(s_{i2} - s_{i1})[\alpha + \beta]$$

$$= -\frac{1}{2}(\alpha + \beta)\sum_{i=1}^{N}(s_{i2} - s_{i1}) = 0 \tag{7}$$

The last equality in equation (7) follows from the fact that expenditure shares add up to 1.
We observe that the result reported here is based on the Törnqvist index. However it is easy to show that this result holds even when other index number formulae are used. Two further results are stated and proved below.

Result 2: Under the set-up considered in equations (1) to (6) and if the Fisher index is used for the purpose of price comparisons across countries and over time then the inconsistency between benchmarks and updating using domestic measures of price changes vanishes if all the commodities considered in the computation exhibit the same price change over time.

Given that all commodities exhibit the same level of price change over time, we can write the prices in period 2 for countries 1 and 2 respectively as:

$$p_{i2}^{t+1} = \beta \cdot p_{i2}^{t} \qquad \textit{and} \qquad p_{i1}^{t+1} = \alpha \cdot p_{i}^{t} \tag{8}$$

Consider the Fisher index which is the geometric mean of the Laspeyres and Paasche indices. Given (8) it follows that the price change from period *t* to *t+1* for counties 1 and 2 are respectively α and β, that is $P_2 = \beta \quad \textit{and} \quad P_1 = \alpha$.

Now we consider the change in the price level for country 2 with country 1 as the reference country. This is given by the ratio:

$$\frac{P_2^{t+1}}{P_2^{t}} = \frac{\left[\dfrac{\sum_{i=1}^{N} p_{i2}^{t+1} q_{i2}^{t+1}}{\sum_{i=1}^{N} p_{i1}^{t+1} q_{i2}^{t+1}} \cdot \dfrac{\sum_{i=1}^{N} p_{i2}^{t+1} q_{i1}^{t+1}}{\sum_{i=1}^{N} p_{i1}^{t+1} q_{i1}^{t+1}}\right]^{0.5}}{\left[\dfrac{\sum_{i=1}^{N} p_{i2}^{t} q_{i2}^{t}}{\sum_{i=1}^{N} p_{i1}^{t} q_{i2}^{t}} \cdot \dfrac{\sum_{i=1}^{N} p_{i2}^{t} q_{i1}^{t}}{\sum_{i=1}^{N} p_{i1}^{t} q_{i1}^{t}}\right]^{0.5}} \tag{9}$$

Substituting (8) into (9) and observing that the expenditure shares remain the same over time, we can show after simple algebraic manipulations that

$$\frac{P_2^{t+1}}{P_2^{t}} = \frac{\left[\dfrac{\sum_{i=1}^{N} p_{i2}^{t+1} q_{i2}^{t+1}}{\sum_{i=1}^{N} p_{i1}^{t+1} q_{i2}^{t+1}} \cdot \dfrac{\sum_{i=1}^{N} p_{i2}^{t+1} q_{i1}^{t+1}}{\sum_{i=1}^{N} p_{i1}^{t+1} q_{i1}^{t+1}}\right]^{0.5}}{\left[\dfrac{\sum_{i=1}^{N} p_{i2}^{t} q_{i2}^{t}}{\sum_{i=1}^{N} p_{i1}^{t} q_{i2}^{t}} \cdot \dfrac{\sum_{i=1}^{N} p_{i2}^{t} q_{i1}^{t}}{\sum_{i=1}^{N} p_{i1}^{t} q_{i1}^{t}}\right]^{0.5}} = \frac{\left[\dfrac{\sum_{i=1}^{N} \beta p_{i2}^{t} q_{i2}^{t+1}}{\sum_{i=1}^{N} \alpha p_{i1}^{t} q_{i2}^{t+1}} \cdot \dfrac{\sum_{i=1}^{N} \beta p_{i2}^{t} q_{i1}^{t+1}}{\sum_{i=1}^{N} \alpha p_{i1}^{t} q_{i1}^{t+1}}\right]^{0.5}}{\left[\dfrac{\sum_{i=1}^{N} p_{i2}^{t} q_{i2}^{t}}{\sum_{i=1}^{N} p_{i1}^{t} q_{i2}^{t}} \cdot \dfrac{\sum_{i=1}^{N} p_{i2}^{t} q_{i1}^{t}}{\sum_{i=1}^{N} p_{i1}^{t} q_{i1}^{t}}\right]^{0.5}} = \frac{\beta}{\alpha} = \frac{P_2}{P_1} \tag{10}$$

Equation (10) implies: $\left(\ln P_2^{t+1} - \ln P_2^{t}\right) - \left(\ln P_2 - \ln P_1\right) = 0$, which in turn implies that there is no inconsistency between the benchmark comparisons and temporal price changes observed in countries 1 and 2.

Now we turn to a more general result that does not depend upon the functional form for the price index. Here a binary index that compares prices in period or country 2 with the base period or reference country 1, denoted by P_{12} , is a function of observed prices and quantities, (p_2, p_1, q_1, q_2) .

We assume that the price index satisfies the following proportionality axioms[3]. The price index is given by a function of prices and quantities observed in the two periods/countries:

$$P_{12} = P(p_2, p_1, q_2, q_1) \tag{11}$$

[3] The axiomatic approach to index numbers is well researched. Comprehensive expositions of the axiomatic approach can be found in Balk (2008) and in ECE-ILO (2010) *Manual on the Consumer Price Index*.

Axiom of Proportionality in prices of current period: The price index $\boldsymbol{P}_{21}$ is said to satisfy this axiom if prices in period 2 are multiplied by a constant λ (> 0) then the index is itself multiplied by λ. That is:

$$\boldsymbol{P}(\lambda \boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{q}_2) = \lambda \boldsymbol{P}(\boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{q}_2) \tag{12}$$

Axiom of Proportionality in prices of base period: The price index $\boldsymbol{P}_{21}$ is said to satisfy this axiom if prices in period 1 are multiplied by a constant λ (>0) then the index is itself multiplied by 1/λ. That is:

$$\boldsymbol{P}(\boldsymbol{p}_2, \lambda \boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{q}_2) = \frac{1}{\lambda} \boldsymbol{P}(\boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{q}_2) \tag{13}$$

The following result provides a sufficient condition for the consistency between benchmarks and temporal price movements.

Result 3: If the price index formula used for comparisons of prices across countries and over time are represented by a generic price index formula $\boldsymbol{P}_{12} = \boldsymbol{P}(\boldsymbol{p}_2, \boldsymbol{p}_1, \boldsymbol{q}_2, \boldsymbol{q}_1)$ and if the index satisfies the axioms of proportionality in current and base period/country prices, then the cross-country price comparisons across two different benchmarks are consistent with relative price movements in the two periods.

The proof follows from the definitions that use notation in equations (1) to (6). We have:

$$\begin{aligned}
\boldsymbol{P}_2^{t+1} &= \boldsymbol{P}(\boldsymbol{p}_2^{t+1}, \boldsymbol{p}_1^{t+1}, \boldsymbol{q}_2^{t+1}, \boldsymbol{q}_1^{t+1}) \\
\boldsymbol{P}_2^{t} &= \boldsymbol{P}(\boldsymbol{p}_2^{t}, \boldsymbol{p}_1^{t}, \boldsymbol{q}_2^{t}, \boldsymbol{q}_1^{t}) \\
\mathrm{P}_2 &= \boldsymbol{P}(\boldsymbol{p}_2^{t+1}, \boldsymbol{p}_2^{t}, \boldsymbol{q}_2^{t+1}, \boldsymbol{q}_2^{t}) \\
\mathrm{P}_1 &= \boldsymbol{P}(\boldsymbol{p}_1^{t+1}, \boldsymbol{p}_1^{t}, \boldsymbol{q}_1^{t+1}, \boldsymbol{q}_1^{t})
\end{aligned} \tag{14}$$

Making using of the fact that $\boldsymbol{p}_2^{t+1} = \beta\, \boldsymbol{p}_2^{t}$ ***and*** $\boldsymbol{p}_1^{t+1} = \alpha\, \boldsymbol{p}_1^{t}$ and using the two axioms stated above, we can show that

$$\frac{\boldsymbol{P}_2^{t+1}}{\boldsymbol{P}_2^{t}} = \frac{\boldsymbol{P}_2}{\boldsymbol{P}_1} = \frac{\beta}{\alpha} \tag{15}$$

Therefore consistency between benchmarks and temporal price movements can be guaranteed in the case where price movements in the countries 1 and 2 are proportional and the index number formula used satisfies the two axioms of proportionality.

- The results stated here provide a sufficient condition but it is not a necessary condition. Further, the result is derived in a very special case.
- We believe that this sufficient condition provides guidance as to the level of disaggregation at which we could extrapolate with minimum inconsistency. The answer according to the result is that the commodity group should be sufficiently homogeneous to exhibit similar price movements over time. In price index compilation, this concept is somewhat similar to commodity groups that underpin elementary indices.
- ***These results suggest that it is best if extrapolation is undertaken at the basic heading level. It is generally expected that the products included in a basic heading are not only homogeneous but they also exhibit similar price level differences across countries and movements over time.***

3. Interpolation of PPPs at the basic heading level - options

We use the following notation in this section. As we focus on the problem of extrapolation/interpolation of PPPs for a given basic heading, we do not use a separate identifier for the basic heading. Let PPP_c^t represent PPP for country c for the year t – all the PPPs are expressed relative to a reference country. We use USA as the reference country. We consider time series for $t = 1, 2, ..., T$ and there are M countries in the comparison with $c = 1, 2, ..., M$.

Data available for interpolation between the years 1 and T

1. We have PPPs, for the basic heading under consideration, for all the countries for the two benchmark years 1 and T. For example, these two represent the benchmarks 2011 and 2017 respectively. These PPPs are denoted by $PPP_c^1 \text{ and } PPP_c^T, c = 1, 2, ..., M$
2. National accounts deflators from each country for each year expressed relative to the year 1 as the base year. Let P_c^t represent the implicit national accounts deflator in country c for the year t with year 1 as the base year.[4] These deflators may be computed using different formulae, e.g. fixed or chain base index numbers computed using Laspeyres or Fisher index numbers. At this stage we do not make any assumptions or restrict the use of the formulae for computation.
3. Expenditure, in national currency units, at current and constant prices. As constant price data are considered as volumes or implicit quantities (for the composite commodity group), we let $E_c^t \text{ and } Q_c^t$ represent, respectively, current and constant price expenditure in country c in period t.[5] Obviously these are linked through the national accounts deflator by the equation

$$P_c^t = \frac{E_c^t}{Q_c^t} \quad for \ c = 1, 2, ..., M \ and \ t = 1, 2, ..., M \tag{16}$$

4. We note here that Q_c^t, expenditure in country c in period t is expressed in the currency units of country c. Therefore, Q_c^t can be compared over time to measure growth rates within each country but these cannot be compared across countries. To facilitate comparison across countries, we convert these into common currency units using purchasing power parities, . thus we have[6]

$$e_c^t = \frac{E_c^t}{PPP_c^t}; \ p_c^t = \frac{P_c^t}{PPP_c^t}; \ \ q_c^t = \frac{Q_c^t}{PPP_c^t} \quad for \ t = 1 and \, T \tag{17}$$

5. Using PPP converted expenditure in (17), we can compute shares of each country in the total aggregate over all the countries in the two periods 1 and T. These shares are defined as:

$$s_c^t = \frac{e_c^t}{\sum_{c=1}^{M} e_c^t} \quad for \ c = 1, 2, ..., M \ and \ t = 1, M \tag{18}$$

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[4] In practice, different countries have different base years in their respective national accounts. We assume, for our purpose, that all the deflators are suitably rebased to have year 1 as the base year.

[5] We endeavor to maintain some similarity in notation with that used in Diewert and Fox (2015), but some differences remain.

[6] We note here that we define these PPP converted aggregates and price deflators only for the years 1 and T for which we have benchmark comparisons. This structure differs from Diewert and Fox (2015) where they have PPPs for all the years in between 1 and T. In a sense, our objective is different from theirs.

6. Using information provided here and following the framework suggested in Diewert and Fox (2015)[7] and Balk, Rambaldi and Rao (2017), we can compute measures of volume growth at the country level using the ratio Q_c^T / Q_c^1 for each country c and for the whole group of M countries using either a Fisher index or Sato-Vartia index.[8]

Now we have established notation and also the type of data available for the purpose of interpolation. We now consider options for interpolation between the two benchmarks years 1 and T or 2011 and 2017 in the case of ICP.

Option 1: Geometric version of PWT interpolation

The Penn World Tables 8.0 onwards have adopted the following interpolation approach in generating PPPs for non-benchmark years in between benchmark years. PWT makes use of a weighted arithmetic average of extrapolated PPP for country c for the year t from the initial benchmark year 1 and retrapolated PPP from the final benchmark year T, where the weights depend on the adjacency of t to the two benchmark years 1 and T. Using the notation in this paper, the PWT extrapolation is given by:

$$PPP_c^t = (1-w^t)\cdot PPP_c^1 \cdot \frac{P_c^t / P_c^1}{P_{USA}^t / P_{USA}^1} + w^t \cdot PPP_c^T \cdot \frac{P_c^t / P_c^T}{P_{USA}^t / P_{USA}^T} \quad c = 1,2,...,M; t = 2,...,T-1 \qquad (19)$$

where $w^t = (t-1)/(T-1)$.

Rationale for the use of (19) is intuitive in that extrapolations closer to benchmark are likely to be more reliable than extrapolations far from the benchmark. For example, in the case of interpolation between 2011 and 2017, it is intuitive that extrapolation from 2011 to 2012 would be more reliable than retrapolation from 2017 to 2012.[9] Weights in (19) linearly decline as we move away from the benchmark, in either direction.

However, the use of arithmetic average has two problems. While the two components that make-up the extrapolation are essentially invariant to the choice of the base country, it is not clear if the arithmetic average is invariant to such a choice. Second, since PPPs are expected to satisfy transitivity in a multiplicative sense, it is important that the extrapolation in (19) makes use of a geometric average instead of arithmetic average. So we suggest the use of geometric version of PWT extrapolation given by:

$$PPP_c^t = \left[PPP_c^1 \cdot \frac{P_c^t / P_c^1}{P_{USA}^t / P_{USA}^1} \right]^{(1-w^t)} \times \left[PPP_c^T \cdot \frac{P_c^t / P_c^T}{P_{USA}^t / P_{USA}^T} \right]^{w^t} \quad for\ c = 1,2,...,M; t = 2,...,T-1 \qquad (20)$$

Diewert and Fox (2015) compare arithmetic average extrapolations from PWT with extrapolations from their method and suggest further work.

Option 2: State-space approach from Rao, Rambaldi and Doran (2010)

[7] The Diewert and Fox (2015) study uses chained-Fisher as they have data for all the years between 2000 and 2012. Here we have data only for the two endf-points.

[8] For a discussion on the use of Fisher and Sato-Vartia indexes and the approach to measuring growth in the group of countries, see Balk, Rambaldi and Rao (2017) where world growth is computed. In contrast, Diewert and Fox (2015 focuses on OECD countries. As both studies use these for illustrating the approach proposed in their respective papers, the approaches can be used here.

[9] This is consistent with the general econometric notion that predictions away from the sample are generally less reliable and have higher standard errors.

The Rao, Rambaldi and Doran (2010) approach adopted to the current problem of filling PPP data gaps in between two benchmarks in the years 1 and T, can be presented in the form of two equations. Here we assume that both benchmarks cover exactly the same list of countries. In the case of Diewert and Fox (2015), the coverage includes all OECD countries through the years 2000 to 2012. However, in the case of ICP the 2011 benchmark covered 177 countries and ICP 2017 is likely to have greater coverage. We will revert to this problem toward the end of this paper.

The basic RRD approach postulates that the observed PPPs in the two benchmark years, 1 and T, are values of the true PPPs with measurement error. Thus

Observation equation: $PPP_c^1 = PPP{*}_c^1 \cdot u_c^1 \;\; and \;\; PPP_c^T = PPP{*}_c^T \cdot u_c^T \quad c = 1, 2, ..., M$ (21)

where $u_c^1 \; and \, u_c^T$ are random disturbance terms with mean 1 and have variances reflecting the reliability of PPPs for each country in each of the two benchmark countries. PPPs for time periods in between 1 and T are obtained by updating PPPs using country-specific deflators over time.

Updating equation:
$$PPP_c^t = \left[PPP_c^1 \cdot \frac{P_c^t / P_c^1}{P_{USA}^t / P_{ISA}^1} \right] \cdot v_c^t \quad c = 1, 2, ..., M; t = 1, 2, ..., T \tag{22}$$

The RRD paper outlines a state-space approach to construct extrapolated PPPs that are consistent with the stochastic framework governing (21) and (22). Estimation of parameters and construction of Kalman Filter and smoother are discussed in detail in Rao, Rambaldi and Rao (2010, 2013). The most pertinent part of the RRD work for the purpose of filling gaps in the years between 1 and T is the result which provides our Option 2.

The RRD approach in this case simplifies to the following option implemented in three stages: (i) First, extrapolate PPPs from benchmark year 1 to all the years using the updating equation in (22).

Forward extrapolation: $\overleftrightarrow{PPP}_c^{t,1} = \left[PPP_c^1 \cdot \frac{P_c^t / P_c^1}{P_{USA}^t / P_{ISA}^1} \right] \cdot v_c^t \quad c = 1, 2, ..., M; t = 1, 2, ..., T$

Second, retrapolate PPPs from benchmark year T backwards to year 1 using the following equation.

Backward extrapolation: $\overleftrightarrow{PPP}_c^{t,T} = \left[PPP_c^T \cdot \frac{P_c^t / P_c^T}{P_{USA}^t / P_{ISA}^T} \right] \cdot v_c^t \quad c = 1, 2, ..., M; t = 1, 2, ..., T$

In the final step, the optimal predictor of PPP for all the years is given by the weighted average of the Forward and backward extrapolation.

When this procedure is implemented, PPPs at the benchmark years are also modified to a small degree. However, assuming that the observation equation in (21) is observed without errors, i.e. the random disturbance term equals 1, then the benchmark PPPs are preserved exactly (also one of the properties of the RRD Method). The extrapolated PPP's for the years 2,…,T-1 are given by the weighted geometric average:

$$PPP_c^t(RRD) = \left(\overleftrightarrow{PPP}_c^{t,1} \right)^{\gamma_c^1} \cdot \left(\overleftrightarrow{PPP}_c^{t,\mathrm{T}} \right)^{\gamma_c^T} \quad for \;\; c = 1, 2, ..., M; t = 2, ..., T-1 \tag{23}$$

where $\gamma_c^1 > 0; \; and \; \gamma_c^T > 0 \;\; such \; that \;\; \gamma_c^1 + \gamma_c^T = 1$. Therefore the RRD approach gives a weighted average of the extrapolations from the two benchmarks with weights depending on the reliability of the benchmark PPPs and the updating equation. The point to note here is that the weights are the same over all the years.

This is in contrast to PWT approach which provides weights for different years but not for the reliability of benchmark data.

Option 3: Combined geometric PWT and RRD Approaches

Here we provide a modification of the PWT approach by taking geometric average and modifying it to accommodate weights from RRTD approach. Thus the proposal is to use:

$$PPP_c^t(RRD\_PWT) = \left(\overleftrightarrow{PPP}_c^{t,1}\right)^{\left(\gamma_c^1 \cdot +1-w_t)\right)/2} \cdot \left(\overleftrightarrow{PPP}_c^{t,\mathrm{T}}\right)^{\left(\gamma_c^T + w_t\right)/2} \quad for\ c = 1,2,...,M; t = 2,...,T-1 \tag{24}$$

It is easy to check that weights used in the geometric average (24) are proper weights in that that are non-negative and add up to 1.

An important feature of Option 3 is that it combines and likely to improve on both the PWT approach and the RRD approaches, each of which have some intuition and technical sophistication. In certain circumstances, it is possible to show that the RRD and PWT approaches are identical. This means that RRD is a more general approach which contains PWT as a special case.

Before we move on to the last option, we note here that the three options considered here make use of only PPPs for the end-point benchmarks at 1 and T and country-specific price deflators. No use is made of expenditure data at the basic heading level which may be considered as a drawback or an advantage depending on the reliability of expenditure data at the basic heading level, especially from developing countries.

Option 4: Diewert and Fox (2015) approach

The following steps are involved in their approach[10].

Step 1: The Diewert and Fox (2015) approach involves the computation of growth rate in the quantity or volume of the whole region. In the current scenario where only two benchmarks are available, their approach requires the use of fixed-base approach to measure growth from period 1 to T. The basic data used are shares of different countries in periods 1 and T defined in equation (18)

$$s_c^t = \frac{e_c^t}{\sum_{c=1}^M e_c^t} \quad for\ c = 1,2,...,M\ \ and\ \ t = 1, M$$

and quantity growth rates in each country over the period which are given by Q_c^T / Q_c^1 . The fixed-base Fisher index of overall growth in the world or group of countries, $c=1,2,...,M$ is then given by:

$$\Gamma_F = \left[\Gamma_L \cdot \Gamma_P\right]^{1/2}\ \ where\, \Gamma_L = \sum_{c=1}^M s_c^1 \cdot \left(\frac{Q_c^T}{Q_c^1}\right);\ \Gamma_P = \left[\sum_{c=1}^M s_c^T \cdot \left(\frac{Q_c^1}{Q_c^T}\right)\right]^{-1} \tag{25}$$

Step2: Construction of interpolated quantities for each time period t for each country c. Interpolated quantities are denoted by $q_{I,c}^t$ where I stands for the fact that these are interpolations. For the two end points

[1010] We adapt the notation of Diewert and Fox (2015) to facilitate ease in reconciling our presentation with their results.

$$q_{I,c}^{1} = s_c^1 \quad and \quad q_{I,c}^{T} = s_c^T \cdot \Gamma_F \qquad (26)$$

The long term implied growth rate for country *c* from (26) is given by: $g_c = \dfrac{q_c^T}{q_c^1}$ (27)

However, during the same period from 1 to T, the observed growth rate is: $G_c = \dfrac{Q_c^T}{Q_c^1}$ (28)

The discrepancy between the country-specific growth rates in (28) are compared with implied growth rates from the world growth in (27). Diewert and Fox (2015) define *the country c proportional annualized discrepancy factor,* α_c , is defined as

$$\alpha_c = \left[\frac{g_c}{G_c}\right]^{1/(T-1)} \qquad (29)$$

Using the discrepancy factor in (29), the interpolated quantity for period *t* is defined as:

$$q_{I,c}^{t} = q_{I,c}^{t-1} \cdot \left[\frac{Q_c^t}{Q_c^{t-1}}\right] \cdot \alpha_c \qquad c = 1,2,3,\ldots,M;\ \mathrm{t} = 1,2,\ldots,\mathrm{T}-1 \qquad (30)$$

Step 3: Computation of interpolated PPPs

Once implied quantities for each period are computed and given that these are already expressed in PPP terms, the interpolated PPPs are given by:

$$PPP_{I,c}^{t} = \frac{E_c^t}{q_{I,c}^t} \quad for\ c = 1,2,\ldots,M\ \ and\ \ t = 2,\ldots,T-1$$

Diewert and Fox (2015) compare the interpolated PPPs from their approach with that of PWT and find significant differences between the two sets of results and they conclude by recommending further work.

The Diewert and Fox (2015) approach and the results depend on a number of factors. (i) Reliability of expenditure data at the basic heading level is not always guaranteed; (ii) In the absence of intermediate data as was the case with their work, the important global/regional growth rate is computed using a fixed based Fisher index. Therefore, results really hinge on the difference between their chained-Fisher growth rates versus and the fixed-base Fisher index. If these diverge, then there is some case for consideration; (iii) In the results reported in Diewert and Fox, there seems to be little difference between Laspeyres, Paasche and Fisher indices. These basically point towards lack of correlation between price and quantity movements. This could be a natural consequence of the fact that the data here are over countries and not commodities within a country. This fact also means that the standard index number theory in this case is probably not applicable as there is no utility maximization and substitutions occurs as in the case of data over commodities. So, some caution is required in any routine application of standard index number methods; (iv) The overall growth of the group of countries can be influenced by a small number of large countries. In the case of the world, growth rates in China and India probably determine the overall growth rate as these are economies with impressive growth rates and these economies are the largest and the third largest economies in the world. So the procedure in equations (29) and (30) adjust national growth rates to bring them in line with growth rates in these few large countries. Something to think about.

4. A simple numerical illustration

We consider a simple case with four countries, labelled A.B,C and D where D is considered the reference county and which has PPP =1 in all periods. We consider four periods 0,1,2,3 where $t = 0$ represents the First Benchmark and $t = 3$ represents the second benchmark. The problem is to fill the gaps in the following table:

Table 1:Tableau of PPPs (Reference country D)

	0 BM1	1	2	3 BM2
A	0.7	?	?	0.8
B	3.4	?	?	4.3
C	89	?	?	91

Deflators used for extrapolation are presented in Table 2

Table 2: Price deflators (expressed relative to reference country D)

	1	2	3 BM2
A	1.04	1.02	1.05
B	1.05	1.05	1.02
C	1	1.02	1.01

In order to implement the RRD approach some reliability measures for the deflators for the three countries in the three periods 1,2, and 3 are needed. We specify the following diagonal matrices Q_1, Q_2, and Q_3 for the three periods.

$$Q_1 = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}; \; Q_1 = \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}; \; Q_3 = \begin{bmatrix} 0.021 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}$$

The weights used in updating based on PWT and the extrapolated series for periods 1 and 2 are given in Table 3 below.

Table 3: PWT Extrapolated series

Country	t=0	t=1					t=2					
	Bench1	F	weight	B	weight	PWT	F	weight	B	weight	PWT	Bench 2
A	0.700	0.728	**0.667**	0.747	**0.333**	**0.734**	0.743	**0.333**	0.762	**0.667**	**0.755**	0.8
B	3.400	3.570	**0.667**	4.015	**0.333**	**3.718**	3.749	**0.333**	4.216	**0.667**	**4.060**	4.3
C	89.000	89.000	**0.667**	88.332	**0.333**	**88.778**	90.780	**0.333**	90.099	**0.667**	**90.326**	91

Here columns for t=0 and t=3 represent PPPs in Benchmark Year 1 and Year 2 respectively. Column F represents forward extrapolation from the benchmark year 1; and B represents backward extrapolation from benchmark year 2. The weights are proportional to the distance from the benchmark used in extrapolation. Columns marked PWT are the final extrapolated series based on a weighted average of the backward and forward extrapolations.

In Table 4, we present predicted series based on our state-space approach. Under the scenario we consider where PPPs in the benchmarks are observed without any errors, then results from RRD (2010) show that predictions for each of the intermediate years is a weighted average of the forward and backward extrapolations. Results are shown below.

Table 4: Extrapolated series based on RRD State-Space approach

					State-space							
Country	t=0	t=1					t=2					
	Bench1	F	weight	B	weight	S	F	weight	B	weight	S	Bench 2
A	0.700	0.728	**0.718**	0.747	**0.282**	**0.732**	0.743	**0.296**	0.762	**0.704**	**0.756**	0.8
B	3.400	3.570	**0.640**	4.015	**0.360**	**3.669**	3.749	**0.353**	4.216	**0.647**	**4.045**	4.3
C	89.000	89.000	**0.750**	88.332	**0.250**	**88.874**	90.780	**0.250**	90.099	**0.750**	**90.269**	91

We note here that the weights used in RRD to combine forward and backward extrapolated series are slightly different from those in PWT. However, the differences are not that large due to the choice of the Q matrices used in the implementation of RRD approach.

5. Conclusions

This short paper canvasses four different options for constructing extrapolated PPPs at the basic heading level for years in between benchmarks. All the options appear plausible and further experimentation with real international comparison data is necessary before a method for use in producing smoothed series is necessary. A numerical illustration implementing the PWT and the state-space approach based on RRD (2010) using a small four-country four-period example. The authors will implement these two approaches as well as the Diewert-Fox approach on data from OECD region.

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