CHAPTER 5

Methods of Aggregation above the Basic Heading Level within Regions

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Chapter 4 describes how the 155 basic heading price parities for each of the *K* countries in a region were constructed for 2005 round of the International Comparison Program (ICP).[1] Once these purchasing power parities (PPPs) have been constructed, aggregate measures of country prices and relative volumes between countries can be constructed using the wide variety of multilateral comparison methods suggested over the years. These aggregate comparisons assume that, in addition to basic heading price parities for each country, national statisticians have provided country expenditures (in their home currencies) for each of the 155 basic heading categories for the reference year 2005. Then the 155 × *K* matrices of basic heading price parities and country expenditures are used to form average price levels across all commodities and relative volume shares for each country.

Many different methods can be used to construct these aggregate purchasing power parities and relative country volumes. P. Hill (2007a, 2007b) surveyed the main methods used in previous rounds of the ICP, as well as other methods that could be used.[2] Only two multilateral methods have been used in previous ICP rounds: (1) the Gini-Éltetö-Köves-Szulc (GEKS) method based on Fisher's bilateral indexes (Fisher 1922), and (2) the Geary-Khamis (GK) method, which is an additive method (Geary 1953; Khamis 1972).

In the 2005 ICP round, aggregate PPPs and relative volumes for countries within each region were constructed for four of the five geographic ICP regions using the GEKS method.[3] However, the Africa region wanted to use an additive method, and so it relied on the relatively new Iklé-Dikhanov-Balk (IDB) additive method for constructing PPPs and relative volumes within the region.[4] The purpose of this chapter is to describe the properties of these three methods—GEKS, GK, and IDB—for making multilateral comparisons between countries in a region.[5] These methods are discussed in sections 5.1, 5.2, and 5.3 of this chapter. The extensive annex to the chapter discusses the properties of the IDB method in more detail because this method is relatively unknown. It may not be of interest, however, to the casual reader.

To discriminate between the various multilateral index number methods suggested for the ICP, it is useful to look at the axiomatic properties of the various methods. Thus section 5.4 lists various axioms or properties or tests that have been suggested for multilateral indexes to see which tests are satisfied by the GEKS, GK, and IDB methods.

Now, a brief comment on the relative merits of the GEKS, GK, and IDB methods is warranted. The GK and IDB methods are *additive methods*—that is, the real final demand of each country can be expressed as the sum of the country's individual basic heading final expenditures. Each real final demand component is weighted by an international price, which is constant across countries. This feature of an additive method is tremendously convenient for users, because the components of final demand can be aggregated consistently across both countries and commodity groups, and so for many purposes it is useful to have available a set of additive international comparisons. However, additive methods are not consistent with the *economic approach* to index number theory (which allows for substitution effects), whereas the GEKS method is consistent. Section 5.5 explains the economic approach and why additive methods are not fully consistent with that approach.

The GEKS multilateral method is fully consistent with the economic approach to making multilateral comparisons. The GEKS approach also has the property that each country in the comparison is treated in a fully symmetric manner—that is, the method is a democratic one. This aspect of GEKS can be considered an advantage of the method. However, from a technical point of view there are some disadvantages to the method in that countries that are at very different stages of development and that face very different relative prices are given the same weight in the method as countries that are at very similar stages of development and face the same structure of relative prices. Bilateral comparisons of countries similar in structure are likely to be much more accurate than comparisons of countries that are very dissimilar. Thus section 5.6 of this chapter introduces an economic approach that builds up a complete multilateral set of comparisons that rests on making bilateral comparisons of countries very similar in structure. Called the minimum spanning tree (MST) method and introduced by R. J. Hill (1999a, 1999b, 2001, 2004, 2009),[6] this method has some advantages over GEKS, and thus it could be considered for use in the next ICP round.

Section 5.7 of this chapter uses the artificial data example in Diewert (1999) to illustrate how the four methods (GEKS, GK, IDB, and MST) differ in a rather extreme numerical example. Two less extreme numerical examples are presented in chapter 6.

5.1 GEKS Method

The GEKS method originated with Gini (1924, 1931), and was independently rediscovered by Éltetö and Köves (1964) and Szulc (1964).

This method is more easily explained by introducing some notation. Let N equal 155 and K be the number of countries in the regional comparison for the reference year. The basic heading PPP for final demand commodity category n and for country k in the region is denoted by $p_n^k > 0$ and the corresponding expenditure (in local currency units) on commodity class n by country k in the reference year by e_n^k for $n = 1, \ldots, N$ and $k = 1, \ldots, K$.[7] Using this information, it is possible to define volumes[8] or implicit quantity levels q_n^k for each basic heading category n and for each country k as the category expenditure deflated by the corresponding basic heading commodity PPP for that country:

$$(5.1) \qquad q_n^k \equiv \frac{e_n^k}{p_n^k}; \quad n = 1, \ldots, N; \; k = 1, \ldots, K.$$

It is useful to define country commodity expenditure shares s_n^k for basic heading class n and country k as

$$(5.2) \qquad s_n^k \equiv \frac{e_n^k}{\sum_{i=1}^{N} e_i^k}; \quad n = 1, \ldots, N;\ k = 1, \ldots, K.$$

Now define country vectors of basic heading PPPs as $p^k \equiv \left[p_1^k, \ldots, p_N^k\right]^T$,[9] country vectors of basic heading volumes as $q^k \equiv \left[q_1^k, \ldots, q_N^k\right]^T$, country expenditure vectors as $e^k \equiv \left[e_1^k, \ldots, e_N^k\right]$, and country expenditure share vectors as $s^k \equiv \left[s_1^k, \ldots, s_N^k\right]^T$ for $k = 1, \ldots, K$.

To define the GEKS parities $P^1, P^2, \ldots, P^K$ between the K countries in the comparison, first define the Fisher (1922) ideal bilateral price index P_F between country j relative to k:[10]

$$(5.3) \qquad P_F(p^k, p^j, q^k, q^j) \equiv \left[\frac{p^j \cdot q^j\, p^j \cdot q^k}{p^k \cdot q^j\, p^k \cdot q^k}\right]^{\frac{1}{2}}; \quad j = 1, \ldots, K;\ k = 1, \ldots, K.$$

Note that the Fisher ideal price index is the geometric mean of the Laspeyres price index between countries j and k, $P_L(p^k, p^j, q^k, q^j) \equiv \dfrac{p^j q^k}{p^k q^k}$,[11] and the Paasche price index, $P_P(p^k, p^j, q^k, q^j) \equiv \dfrac{p^j q^j}{p^k q^j}$.[12] Various justifications for the use of the Fisher ideal index in the bilateral context have been made by Diewert (1976; 1992; 2002, 569) and others.[13] The Fisher index can be justified from the point of view of finding the "best" symmetric average of the Paasche and Laspeyres indexes, or from the point of view of the axiomatic or test approach to index number theory, or from the viewpoint of the economic approach to index number theory.[14]

The aggregate PPP for country j, P^j, is defined as

$$(5.4) \qquad P^j \equiv \prod_{k=1}^{K}\left[P_F(p^k, p^j, q^k, q^j)\right]^{\frac{1}{K}}; \quad j = 1, \ldots, K.$$

Once the GEKS P^j's have been defined by (5.4), the corresponding GEKS country real expenditures or volumes Q^j can be defined as the country expenditures $p^j q^j$ in the reference year divided by the corresponding GEKS purchasing power parity P^j:

$$(5.5) \qquad Q^j \equiv \frac{p^j q^j}{P^j}; \quad j = 1, \ldots, K.$$

If all of the P^j defined by (5.4) are divided by a positive number, say α, then all of the Q^j defined by (5.5) can be multiplied by this same α without materially changing the GEKS multilateral method. If country 1 is chosen as the numeraire country in the region, then set α equal to P^1 defined by (5.4) for $j = 1$, and the resulting price level P^j is interpreted as the number of units of country j's currency required to purchase one unit of country 1's currency and receive an equivalent amount of utility. The rescaled Q^j is interpreted as the volume of final demand of country j in the currency units of country 1.

It is also possible to normalize the aggregate real expenditure of each country in common units (Q^k) by dividing each Q^k by the sum $\sum_{j=1}^{K} Q^j$ in order to express each country's real expenditure or real final demand as a fraction or share of total regional real expenditure—that is, define country k's share of regional real expenditures, S^k, as follows:[15]

$$(5.6) \qquad S^k \equiv \frac{Q^k}{\sum_{j=1}^{K} Q^j}; \quad k = 1, \ldots, K.$$

The country shares of regional real final demand, S^k, remain unchanged after rescaling the PPPs by the scalar α.[16]

5.2 Geary-Khamis Method

The method was suggested by Geary (1958), and Khamis (1972) showed that the equations that define the method have a positive solution under certain conditions.

The GK system of equations involves K country price levels or PPPs, $P^1, \ldots, P^K$, and N international basic heading commodity reference prices, $\pi_1, \ldots, \pi_N$. The equations that determine these unknowns (up to a scalar multiple) are

$$(5.7) \qquad \pi_n = \sum_{k=1}^{K} \left[\frac{q_n^k}{\sum_{j=1}^{K} q_n^j} \right] \left[\frac{p_n^k}{P^k} \right]; \quad n = 1, \ldots, N$$

and

$$(5.8) \qquad P^k = \frac{p^k q^k}{\pi q^k}; \quad k = 1, \ldots, K$$

where $\pi \equiv [\pi_1, \ldots, \pi_N]$ is the vector of GK regional average reference prices. If a solution to equations (5.7) and (5.8) exists, then if all of the country parities P^k are multiplied by a positive scalar, say λ, and all of the reference prices π_n are divided by the same λ, another solution to (5.7) and (5.8) is obtained. Thus π_n and P^k are determined only up to a scalar multiple and an additional normalization is required such as

$$(5.9) \qquad P^1 = 1$$

in order to uniquely determine the parities. It can also be shown that only $N + K - 1$ of the N equations in (5.7) and (5.8) are independent. Once the parities P^k have been determined, the real expenditure or volume for country k, Q^k, can be defined as country k's nominal value of final demand in domestic currency units, $p^k q^k$, divided by its PPP, P^k:

$$(5.10) \qquad Q^k = \frac{p^k q^k}{P^k}; \quad k = 1, \ldots, K,$$

which equals πq^k using (5.8).

The second set of equations in (5.10) characterizes an additive method[17]—that is, the real final demand of each country can be expressed as a sum of the country's individual basic heading final demand volume components, where each real final demand component is weighted by an *international price* that is constant across countries.

Finally, if equations (5.10) are substituted into the regional share equations, (5.6), then country k's share of regional real expenditures is

$$(5.11) \qquad S^k = \frac{\pi q^k}{\pi q}; \quad k = 1, \ldots, K$$

where the region's total volume vector q is defined as the sum of the country volume vectors:

$$(5.12) \qquad q \equiv \sum_{j=1}^{K} q^j$$

Equations (5.10) show the convenience of having an additive multilateral comparison method: when country outputs are valued at the international reference prices, values are additive across both countries and commodities. However, additive multilateral methods are not consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two; see section 5.5. In addition, equations (5.7) reveal that large countries will have a larger contribution to determination of the international prices π_n, and thus these international prices will be much more representative for the largest countries in the comparison than for the smaller ones.[18] This observation leads to the next method for making multilateral comparisons: an additive method that does not suffer from this problem of big countries having an undue influence on the comparison.

5.3 Iklé-Dikhanov-Balk Method

Iklé (1972, 202–4) proposed this method in a very indirect way; Dikhanov (1994, 1997) suggested the much clearer system described here—see equations (5.13) and (5.14); and Balk (1996, 207–8) provided the first existence proof. The equations produced by Dikhanov (1994, 9–12) that are the counterparts to the GK equations (5.7) and (5.8) are

$$(5.13) \qquad \pi_n = \left[\frac{\sum_{k=1}^{K} s_n^k \left[\frac{p_n^k}{P^k} \right]^{-1}}{\sum_{j=1}^{K} s_n^j} \right]^{-1}; \quad n = 1, \ldots, N$$

and

$$(5.14) \qquad P^k = \left[\sum_{n=1}^{N} s_n^k \left[\frac{p_n^k}{\pi_n} \right]^{-1} \right]^{-1}; \quad k = 1, \ldots, K$$

where the country expenditure shares s_n^k are defined by (5.2).

As in the GK method, equations (5.13) and (5.14) involve the K country price levels or PPPs, $P^1, \ldots, P^K$, and N international commodity reference prices, $\pi_1, \ldots, \pi_N$. Equations (5.13) indicate that the n-th international price, π_n, is a share-weighted harmonic mean of the country k basic heading PPPs for commodity n, p_n^k, deflated by country k's overall PPP, P^k. The country k share weights for commodity n, s_n^k, do not sum (over countries k) to unity, but when s_n^k is divided by $\sum_{j=1}^{K} s_n^j$, the resulting normalized shares do sum (over countries k) to unity. Thus equations (5.13) are similar to the GK equations (5.7), except that now a harmonic mean of the deflated basic heading commodity n "prices," $\frac{p_n^k}{P^k}$, is used in place of the old arithmetic mean. Also, in the GK equations country k's volume share of commodity group n in the region, $\frac{q_n^k}{\sum_{j=1}^{K} q_n^j}$, was used as a weighting factor (and thus large countries had a large influence in forming these weights), but now the weights involve country *expenditure* shares, and so each country in the region has a more equal influence in forming the weighted average. Equations (5.14) indicate that P^k, the PPP for country k, is equal to a weighted harmonic mean of the country k basic heading PPPs, p_n^k, deflated by the international price for commodity group n, π_n, where the summation is over commodities n instead of over countries k as in equations (5.13). The share weights in the harmonic means defined by (5.14), s_n^k, sum to one when the summation is over n, and so there is no need to normalize these weights as was the case for equations (5.13).

If a solution to equations (5.13) and (5.14) exists, then multiplication of all of the country parities P^k by a positive scalar λ and division all of the reference prices π_n by the same λ will lead to another solution to (5.13) and (5.14). Thus π_n and P^k are determined only up to a scalar multiple, and an additional normalization is required such as (5.9), $P^1 = 1$.

Although the IDB equations (5.14) do not appear to be related very closely to the corresponding GK equations (5.8), these two sets of equations are actually the same system. To see this, note that the country k expenditure share for commodity group n, s_n^k, is represented by

$$(5.15) \qquad s_n^k = \frac{p_n^k q_n^k}{p^k q^k}; \quad n = 1, \ldots, N; \; k = 1, \ldots, K.$$

Now substitute equations (5.15) into equations (5.14) to obtain

$$(5.16) \qquad P^k = \frac{1}{\sum_{n=1}^{N} s_n^k \left[\frac{P_n^k}{\pi_n}\right]^{-1}}; \quad k = 1, \ldots, K$$

$$= \frac{1}{\sum_{n=1}^{N} \left[\frac{p_n^k q_n^k}{p^k q^k}\right]\left[\frac{\pi_n}{p_n^k}\right]}$$

$$= \frac{p^k q^k}{\sum_{n=1}^{N} \pi_n q_n^k}$$

$$= \frac{p^k q^k}{\pi q^k}.$$

Thus equations (5.14) are equivalent to equations (5.8), and the IDB system is an additive system—that is, equations (5.10)–(5.12) can be applied to the present method just as they were applied to the GK method for making international comparisons.[19]

The annex to this chapter demonstrates several different ways of representing the IDB system of parities, and provides proofs of the existence and uniqueness of the IDB parities. Effective methods for obtaining solutions to the system of equations (5.13) and (5.14) (with a normalization) are presented as well.

As noted at the outset of this chapter, the IDB method was used by the Africa region to construct regional aggregates. This method appears to be an "improvement" over the GK method in that large countries no longer have a dominant influence on the determination of the international reference prices π_n, and so if an additive method that has more democratic reference prices is required, IDB appears to be "better" than GK. In addition, Deaton and Heston (2010) have shown empirically that the IDB method generates aggregate PPPs that are much closer to the GEKS PPPs than are the GK PPPs, using the 2005 ICP data. However, in section 5.5 it is shown that if one takes the economic approach to index number comparisons, then any additive multilateral method will be subject to some substitution bias.

For many users, however, possible substitution bias in the multilateral method is not an important issue: these users want an additive multilateral method so they can aggregate in a consistent fashion across countries and commodity groups. For these users, it may be useful to look at the axiomatic properties of the GK and IDB multilateral methods in order to determine a preference for one or the other of these additive methods. Thus in the next section, various multilateral axioms or tests are listed, and the consistency of GK, IDB, and GEKS with these axioms is determined.

5.4 Test or Axiomatic Approach to Making Multilateral Comparisons

Balk (1996) proposed a system of nine axioms for multilateral methods based on the earlier work of Diewert (1988).[20] Diewert (1999, 16–20) further refined his set of axioms, and this section lists 11 of the 13 "reasonable" axioms he proposed for a multilateral system. The notation used here is $P \equiv [p^1, \ldots, p^K]$ signifies an $N \times K$ matrix for which domestic basic heading parities (or "price" vectors) $p^1, \ldots, p^K$ serve as its K columns, and $Q \equiv [q^1, \ldots, q^K]$ signifies an $N \times K$ matrix for which country basic heading volumes (or "quantity" vectors) $q^1, \ldots, q^K$ serve as its K columns.

Any multilateral method applied to K countries in the comparison determines the country aggregate volumes, $Q^1, \ldots, Q^K$, along with the corresponding country PPPs, $P^1, \ldots, P^K$. The country volumes Q^k can be regarded as functions of the data matrices P and Q so that the country volumes can be written as functions of the two data matrices P and Q—that is, the functions $Q^k(P, Q)$ for $k = 1, \ldots, K$. Once the functions $Q^k(P,Q)$ have been determined by the multilateral method, then country k's share of total regional real expenditures, $S^k(P, Q)$, can be defined as

$$(5.17) \qquad S^k(P, Q) \equiv \frac{Q^k(P, Q)}{[Q^1(P, Q) + \cdots + Q^K(P, Q)]}; \quad k = 1, \ldots, K.$$

Both Balk (1996, 2008) and Diewert (1988, 1999) used the system of regional share equations $S^k(P,Q)$ as the basis for their axioms.

What follows are 11 of Diewert's 13 tests or axioms for a multilateral share system, $S^1(P, Q)$, $\ldots$, $S^K(P, Q)$ (Diewert 1999, 16–20).[21] It is assumed that the two data matrices, P and Q, satisfy some mild regularity conditions, which are listed in section 5A.1.1 in the annex to this chapter. In keeping with the literature on test approaches to index number theory, components of the data matrix Q will be referred to as "quantities" (when they are actually basic heading volumes by commodity group and country) and the components of data matrix P will be referred to as "prices" (when they are actually basic heading PPPs by commodity group and by country).

T1: *Share Test:* There exist K continuous, positive functions, $S^k(P, Q)$, $k = 1, \ldots, K$, such that $\sum_{k=1}^{K} S^k(P, Q) = 1$ for all P, Q in the appropriate domain of definition.

This is a very mild test of consistency for the multilateral system.

T2: *Proportional Quantities Test:* Suppose that $q^k = \beta_k q$ for some $q >> 0_N$ and $\beta_k > 0$ for $k = 1, \ldots,$ K, with $\sum_{k=1}^{K} \beta_k = 1$. Then $S^k(P, Q) = \beta_k$ for $k = 1, \ldots, K$.

This test says that if the quantity vector for country k, q^k, is equal to the positive fraction β_k times the total regional quantity vector q, then that country's share of regional real expenditures, $S^k(P, Q)$, should equal that same fraction β_k. Note that this condition is to hold no matter what P is.

T3: *Proportional Prices Test:* Suppose that $p^k = \alpha_k p$ for $p >> 0_N$ and $\alpha_k > 0$ for $k = 1, \ldots, K$. Then $S^k(P, Q) = \dfrac{pq^k}{\left[p\sum_{i=1}^{K} q^i\right]}$ for $k = 1, \ldots, K$.

This test says the following: suppose that all of the country price vectors p^k are proportional to a common "price"vector p. Then the country k share of regional real expenditure, $S^k(P, Q)$, is equal to the value of its quantity vector, valued at the common prices p, $pq^k \equiv \sum_{n=1}^{N} p_n q_n^k$, divided by the regional value of real expenditures, also valued at the common prices p, $p\sum_{i=1}^{K} q^i$. Thus if prices are proportional to a common set of prices p across all countries, then these prices p can act as a set of reference international prices and the real expenditure volume of country k, Q^k, should equal pq^k up to a normalizing factor.

T4: *Commensurability Test:* Let $\delta_n > 0$ for $n = 1, \ldots, N$, and let Δ denote the $N \times N$ diagonal matrix with the δ_n on the main diagonal. Then $S^k(\Delta P, \Delta^{-1} Q) = S^k(P, Q)$ for $k = 1, \ldots, K$.

This test implies that the country shares $S^k(P, Q)$ are invariant to changes in the units of measurement. This is a standard (but important) test in the axiomatic approach to index number theory that dates back to Fisher (1922, 420).

T5: *Commodity Reversal Test:* Let $\prod$ denote an $N \times N$ permutation matrix. Then $S^k(\prod P, \prod Q) =$ $S^k(P, Q)$ for $k = 1, \ldots, K$.

This test says that the ordering of the N commodity groups should not affect each country's share of regional real expenditure. This test also dates back to Fisher (1922, 63) in the context of bilateral index number formulas.

T6: *Multilateral Country Reversal Test:* Let $S(P, Q)$ denote a K dimensional column vector that has the country shares $S^1(P, Q), \ldots, S^K(P, Q)$ as components, and let $\prod^*$ be a $K \times K$ permutation matrix. Then $S(P\prod^*, Q\prod^*) = S(P, Q)\prod^*$.

This test implies that countries are treated in a symmetric manner—that is, the country shares of world output are not affected by a reordering of the countries. The next two tests are homogeneity tests.

T7: *Monetary Units Test:* Let $\alpha_k > 0$ for $k = 1, \ldots, K$. Then $S^k(\alpha_1 p^1, \ldots, \alpha_K p^K, Q) = S^k(p^1, \ldots, p^K, Q) =$ $S^k(P, Q)$ for $k = 1, \ldots, K$.

This test implies that the absolute scale of domestic prices in each country does not affect each country's share of world output—that is, only relative prices within each country affect the multilateral volume parities.

T8: *Homogeneity in Quantities Test:* For $i = 1, \dots, K$, let $\lambda_i > 0$ and let j denote another country not equal to country i. Then $\dfrac{S^i(P, q^1, \dots, \lambda_i q^i, \dots, q^K)}{S^j(P, q^1, \dots, \lambda_i q^i, \dots, q^K)} = \dfrac{\lambda_i S^i(P, q^1, \dots, q^i, \dots, q^K)}{S^j(P, q^1, \dots, q^i, \dots, q^K)} = \dfrac{\lambda_i S^i(P, Q)}{S^j(P, Q)}$.

This test is equivalent to saying that the volume share of country i relative to country j is linearly homogeneous in the components of the country i quantity vector q^i.

T9: *Monotonicity in Quantities Test:* For each k, $S^k(P, q^1, \dots, q^{k-1}, q^k, q^{k+1}, \dots, q^K) = S^k(P, Q)$ increasing in the components of q^k.

This test says that country k's share of world output increases as any component of the country k quantity vector q^k increases.

T10: *Country Partitioning Test:* Let A be a strict subset of the indexes $(1, 2, \dots, K)$ with at least two members. Suppose that for each $i \in A$, $p^i = \alpha_i p^a$ for $\alpha_i > 0$, $p^a >> 0_N$, and $q^i = \beta_i q^a$ for $\beta_i > 0$, $q^a >> 0_N$ with $\sum_{i \in A} \beta_i = 1$. Denote the subset of $\{1, 2, \dots, K\}$ that does not belong to $A \times B$, and denote the matrices of country price and quantity vectors that belong to $B \times P^b$ and Q^b, respectively. Then, (i) for $i \in A, j \in A$, $\dfrac{S^i(P, Q)}{S^j(P, Q)} = \dfrac{\beta_i}{\beta_j}$, and (ii) for $i \in B$, $S^i(P, Q) = S^{i*}(p^a, P^b, q^a, Q^b)$, where $S^{k*}(p^a, P^b, q^a, Q^b)$ is the system of share functions that is obtained by adding the group A aggregate price and quantity vectors, p^a and q^a respectively, to the group B price and quantity data, P^b, Q^b.

Thus if the aggregate quantity vector for the countries in group A were distributed proportionally among its members (using the weights β_i) and each group A country faced prices that were proportional to p^a, then part (i) of T10 requires that the group A share functions reflect this proportional allocation. Part (ii) of T10 requires that the group B share functions are equal to the same values no matter whether one uses the original share system or a new share system where all of the group A countries have been aggregated up into the single country that has the price vector p^a and the group A aggregate quantity vector q^a. Conversely, this test can be viewed as a consistency in aggregation test if a single group A country is partitioned into a group of smaller countries.

T11: *Additivity Test:* For each set of price and quantity data, P, Q, belonging to the appropriate domain of definition, there exists a set of positive world reference prices $\pi >> 0_N$ such that $S^k(P, Q) = \dfrac{\pi q^k}{\left[\pi \sum_{i=1}^{K} q^i\right]}$ for $k = 1, \dots, K$.

Thus if the multilateral system satisfies test T11, then it is an additive method because the real expenditure Q^k of each country k is proportional to the inner product of the vector of international prices π with the country k vector of commodity volumes (or "quantities"), q^k.

It is useful to contrast the axiomatic properties of the IDB method with the other additive method that has been used in the ICP, the GK system. Based on the results in Diewert (1999) on the GK system and the results in the annex to this chapter on the IDB system, it can be seen that both methods satisfy tests T1–T7 and T11 and that both methods fail T9, the monotonicity in quantities test. Thus the tests that discriminate between the two methods are T8 and T10: the IDB multilateral system passes T8, the homogeneity test, and fails T10, the country partitioning test, and vice versa for the GK system.[22] There has been more discussion of test T10 than test T8. On the one hand, proponents of the GK system like its good aggregation (across countries) properties, and the fact that big countries have more influence on the determination of the world reference price vector π is regarded as a reasonable price to pay for these "good" aggregation properties.[23] On the other hand, proponents of the IDB method like the fact that the world reference prices are more democratically determined (large countries play a smaller role in determination of the vector of international prices π), and they place less weight on good aggregation properties. Also, from evidence presented by Deaton and Heston (2010) using the 2005 ICP database, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method has the advantage that it is an additive method that does not depart too far from the parities generated by the GEKS method.

Diewert (1999, 18) showed that the GEKS system (using the Fisher ideal index as the basic building block) passed tests T1–T9 but failed T10, the country partitioning test, and T11, the additivity test. Thus all three of the multilateral methods considered thus far fail 2 out of the 11 tests.

At this point, it is difficult to unambiguously recommend any one of the three multilateral methods over the other two. The following section considers an economic approach to making multilateral comparisons that may help in evaluating the three methods.

5.5 Additive Multilateral Methods and the Economic Approach to Making Index Number Comparisons

It is useful to begin this section by reviewing the essential assumptions for the economic approach to index number theory:

- Purchasers have preferences over alternative bundles of the goods and services they purchase.
- As a result, they buy more of the things that have gone down in relative price and fewer of the things that have gone up in relative price.

This kind of substitution behavior is well documented, and therefore it is useful to attempt to take it into account when doing international comparisons.

The economic approach to index number theory does take substitution behavior into account. This approach was developed by Diewert in both the bilateral context (1976)[24] and the multilateral context (1999). This theory works as follows:

- Assume that all purchasers have the same preferences over commodities and that these preferences can be represented by a homogeneous utility function.
- Find a functional form that can approximate preferences to the second order[25] and has an exact index number formula associated with it. The resulting index number formula is called a *superlative index number formula*.[26]
- Use the superlative index number formula in a bilateral context so that the real output of every country in the region can be compared with the real output of a numeraire country using this formula. The resulting relative volumes are dependent on the choice of the numeraire country.
- Take the geometric average of all K sets of relative volumes using each country in the region as the numeraire country. This set of average relative volumes can then be converted into regional shares as in section 5.1. The resulting method is called a *superlative multilateral method* (see Diewert 1999, 22).

It turns out that the GEKS method discussed earlier in section 5.1 is a superlative multilateral method (see Diewert 1999, 36). The GEKS method also has quite good axiomatic properties as was shown in section 5.4.

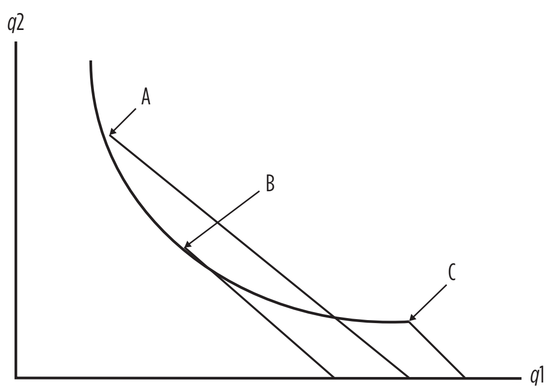
In view of the importance of the GEKS multilateral method, it is worth explaining that the GEKS volume parities can be obtained by alternative methods.

In the first method, described by Deaton and Heston (2010), the GEKS parities can be obtained by using a least squares minimization problem (Gini 1924) that will essentially make an $K \times K$ matrix of bilateral Fisher volume parities that are not transitive into a best-fitting set of transitive parities. In the second method for deriving the GEKS parities, implicitly explained earlier, the parities are obtained by picking any country as the base country and then using the Fisher bilateral quantity index to form the real final demand volume of every country relative to the chosen base country. This process gives estimated volumes for all countries in the comparison relative to the chosen base country. Then this process is repeated, choosing each country in turn as the base country, which leads to K sets of relative volume estimates. The final step for obtaining the GEKS relative volumes is to take the geometric mean of all of the K base country dependent sets of parities.

The problem with an additive multilateral method (from the perspective of the economic approach) if the number of countries in the region is greater than two can now be explained with the help of a diagram (figure 5.1).

The solid curved line in figure 5.1 represents an indifference curve for purchasers of the two goods under consideration. The consumption vectors of countries A, B, and C are all on the same indifference curve, and thus the multilateral method should show the same volume for the three countries. If one uses the relative prices that country B faces as "world" reference prices in an additive method, then country B has the lowest volume or real consumption, followed by country A; country C has the highest volume. But they all have equal volumes! It is possible to devise an additive method that will make the volumes of any two countries equal, but it is not possible to devise an additive method that will equalize the volumes for all three countries. On the other hand, the common indifference curve in figure 5.1 can be approximated reasonably well by a flexible functional form that has a corresponding exact index number formula (such as the Fisher index), and thus a GEKS method that used the Fisher bilateral index as a basic building block would give the right answer to a reasonable degree of approximation. The bottom line is that an additive multilateral method is not really consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two.[27]

FIGURE 5.1 Indifference Curve for Two Products, Three Countries



Source: Based on Marris 1984, 52, and Diewert 1999, 48–50.

Although additive multilateral methods have their problems in that they are not consistent with substitution in the face of changing relative prices, the economic approach as explained earlier is not without its problems. Two important criticisms of the economic approach are (1) the assumption that all final purchasers have the same preferences over different baskets of final demand purchases is suspect, and (2) the assumption that preferences are homothetic—that is, can be represented by a linearly homogeneous utility function—is also suspect.

The second criticism of the economic approach to multilateral comparisons based on superlative bilateral index number formulas has been discussed in the recent literature on international comparisons, and some brief comments on this literature are in order here.

An important recent development is Neary's GAIA multilateral system. It can be described as a consumer theory–consistent version of the GK system, which allows for nonhomothetic preferences on the part of final demanders (Neary 2004). Deaton and Heston (2010) point out that a weakness of the Neary multilateral system is that it uses a single set of relative prices to value consumption or the gross domestic product (GDP) in all countries, no matter how different are the actual relative prices in each country. This problem was also noted by Feenstra, Ma, and Rao (2009), who generalized Neary's framework to work with two sets of cross-sectional data in order to estimate preferences.[28] They also experimented with alternative sets of reference prices. In their discussion of Feenstra, Mao, and Rao, Barnett, Diewert, and Zellner (2009) noted that a natural generalization of their model would be use of a set of reference prices that would be representative for each country in the comparison. Using representative prices for each country would lead to K sets of relative volumes, and in the end these country-specific parities could be averaged, just as the GEKS method averages country-specific parities. Barnett, Diewert, and Zellner conjectured that this geometric average of the country estimates would probably be close to GEKS estimates based on traditional multilateral index number theory, which, of course, does not use econometrics. It remains to be seen if econometric approaches to the multilateral index number problem can be reconciled with superlative multilateral methods.[29]

The next section describes another economic approach to constructing multilateral comparisons—a method that is based on linking countries that have similar economic structures.

5.6 Minimum Spanning Tree Method for Making Multilateral Comparisons

Recall that the Fisher ideal quantity index can be used to construct real volumes for all K countries in the comparison, using one country as the base country. Thus as each country is used as the base country, K sets of relative volumes are obtained. The GEKS multilateral method treats each country's set of relative volumes as being equally valid, and thus an averaging of the parities is appropriate under this hypothesis. The method is therefore "democratic" in that each bilateral index number comparison between any two countries receives the same weight in the overall method. However, not all bilateral comparisons of volume between two countries are equally accurate. On the one hand, if the relative prices in countries A and B are very similar, then the Paasche and Laspeyres volume or quantity indexes will be very close, and so it is likely that the "true" volume comparison between these two countries (using the economic approach to index number theory) will be very close to the Fisher volume comparison. On the other hand, if the structure of relative prices in the two countries is very different, then it is likely that the structure of relative quantities in the two countries will also be different. Therefore, the Paasche and Laspeyres quantity indexes will likely differ considerably, and it is no longer so certain that the Fisher quantity index will be close to the "true" volume comparison. These considerations suggest that a more accurate set of world product shares could be constructed if initially a bilateral comparison was made between the two countries that had the most similar relative price structures.[30] At the next stage of the comparison, one could look for a third country that has a relative price structure most similar to the those of the first two countries and link in this third country to the comparisons of volume between the first two countries, and so on. At the end of this procedure, a minimum spanning tree would be constructed, which is a path between all countries that minimizes the sum of the relative price dissimilarity measures.[31] The conclusion is that similarity linking[32] using Fisher ideal quantity indexes as the bilateral links is an alternative to the GEKS method, which has some advantages over it.[33] Both methods are consistent with the economic approach to index number theory.

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two countries. Various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981); Kravis, Heston, and Summers (1982, 104–6); Aten and Heston (2009); Diewert (2009); R. J. Hill (1997, 2009); and Sergeev (2001, 2009). A few of these suggested measures of dissimilarity will now be discussed.

Suppose one wishes to compare the similarities in the structure of relative prices for two countries, 1 and 2. They have the strictly positive basic heading PPP vectors p^k and the basic heading volume vectors q^k for $k = 1, 2$. For convenience of exposition, in remainder of this section the PPP vector p^k is referred to as a "price" vector and the volume vector q^k as a "quantity" vector. A *dissimilarity index*, $\Delta(p^1, p^2, q^1, q^2)$, is a function defined over the "price" and "quantity" data pertaining to the two countries, p^1, p^2, q^1, q^2, which indicates how similar or dissimilar the structure of relative prices is in the two countries being considered. If the two price vectors are proportional so that the relative prices in the two countries are equal, then one wants the dissimilarity index to equal its minimum possible value, zero—that is, one wants $\Delta(p^1, p^2, q^1, q^2)$ to equal zero if $p^2 = \lambda p^1$ for any positive scalar λ. If the price vectors are not proportional, then one wants the dissimilarity measure to be positive.[34] Thus the larger is $\Delta(p^1, p^2, q^1, q^2)$, the more dissimilar is the structure of relative prices between the two countries.

The first measure of dissimilarity in relative price structures was suggested by Kravis, Heston, and Summers (1982, 105)[35] and R. J. Hill (1999a, 1999b, 2001, 2004). It is essentially a normalization of the relative spread between the Paasche and Laspeyres price indexes, and so it is known as the Paasche-Laspeyres spread (PLS) relative price dissimilarity measure, $\Delta_{\text{PLS}}(p^1, p^2, q^1, q^2)$, for which

$$(5.18) \qquad \Delta_{\text{PLS}}(p^1, p^2, q^1, q^2) \equiv \max\left[\frac{P_L}{P_P}, \frac{P_P}{P_L}\right] - 1 \geq 0$$

where $P_L \equiv \dfrac{p^2 q^1}{p^1 q^1}$ and $\dfrac{p^2 q^2}{p^1 q^2}$. Thus if $P_L = P_P$, the dissimilarity measure defined by (5.18) takes on its minimum value of zero. Because P_L differs more markedly from P_P, the dissimilarity measure increases and the relative price structures are regarded as being increasingly dissimilar. Diewert (2009, 184) pointed out a major problem with this measure of relative price dissimilarity; it is possible for P_L to equal P_P, and yet p^2 could be very far from being proportional to p^1. The following two measures of dissimilarity do not suffer from this problem.

Diewert (2009, 207) suggested the following measure of relative price similarity, the weighted log quadratic (WLQ) measure of relative price dissimilarity, $\Delta_{\text{WLQ}}(p^1, p^2, q^1, q^2)$, for which

$$(5.19) \quad \Delta_{\text{WLQ}}(p^1, p^2, q^1, q^2) \equiv \sum_{n=1}^{N}\left(\frac{1}{2}\right)\left(s_n^1 + s_n^2\right)\left[\ln\left(\frac{p_n^2}{p_n^2\, P_F(p^1, p^2, q^1, q^2)}\right)\right]$$

where $P_F(p^1, p^2, q^1, q^2) \equiv \left[\frac{p^2 \cdot q^1 p^2 \cdot q^2}{p^1 \cdot q^1 p^1 \cdot q^2}\right]^{\frac{1}{2}}$ is the Fisher ideal price index between countries 2 and 1, and $s_n^c \equiv \frac{p_n^c q_n^c}{p^c q^c}$ is the country c expenditure share on commodity n for $c = 1, 2$ and $n = 1, \ldots, N$.

There is a problem with the dissimilarity measure defined by (5.19) if for some commodity group n either p_n^1 or p_n^2 equals zero (or both prices equal zero), because in these cases the measure can become infinite.[36] If both prices are zero, then commodity group n is irrelevant for both countries and the n-th term in the summation in (5.19) can be dropped. In the case in which one of the prices, say p_n^1, equals zero but the other price p_n^2 is positive, then it would be useful to have an imputed PPP or "price" for commodity group n in country 1 that will make the final demand volume for that commodity group equal to zero. This reservation price, say p_n^{1*}, could be approximated by simply setting p_n^{1*} equal to $\frac{p_n^2}{P_F(p^1, p^2, q^1, q^2)}$. If p_n^1 equal to zero in (5.19) is replaced by this imputed price p_n^{1*}, then $\frac{p_n^2}{p_n^{1*} P_F(p^1, p^2, q^1, q^2)}$ is equal to one, and the n-th term on the right-hand side of (5.19) vanishes. Similarly, in the case in which p_n^2 equals zero but the other price p_n^1 is positive, then set the reservation price for the n-th commodity group in country 2, say p_n^{2*}, equal to $p_n^1 P_F(p^1, p^2, q^1, q^2)$. If the zero price p_n^2 in (5.19) is replaced by the imputed price p_n^{2*}, then $\frac{p_n^{2*}}{p_n^1 P_F(p^1, p^2, q^1, q^2)}$ is equal to one, and the n-th term on the right-hand side of (5.19) also vanishes in this case. Thus if there is a zero "price" for either country for commodity group n, then the earlier convention for constructing an imputed price for the zero price leads to dropping the n-th term on the right-hand side of (5.19).[37]

If prices are proportional for the two countries so that $p^2 = \lambda p^1$ for some positive scalar λ, then $P_F(p^1, p^2, q^1, q^2) = \lambda$, and the measure of relative price dissimilarity $\Delta_{PLQ}(p^1, p^2, q^1, q^2)$ defined by (5.19) will equal its minimum of zero. Thus the smaller is $\Delta_{PLQ}(p^1, p^2, q^1, q^2)$, the more similar is the structure of relative prices in the two countries.

The method of spatial linking using the relative price dissimilarity measure defined by (5.19) is illustrated in the next section.[38] The shares generated by the minimum spanning tree—not the GEKS country shares defined by (5.6) in section 5.1—are used to link all of the countries in the comparison.

Diewert (2009, 208) also suggested the following measure of relative price similarity, the weighted asymptotically quadratic (WAQ) measure of relative price dissimilarity, $\Delta_{WAQ}(p^1, p^2, q^1, q^2)$, for which

$$(5.20) \qquad \Delta_{WAQ}(p^1, p^2, q^1, q^2) \equiv \sum_{n=1}^{N} \left(\frac{1}{2}\right)\left(s_n^1 + s_n^2\right)\left\{\left[\frac{p_n^2}{p_n^1 P_F(p^1, p^2, q^1, q^2)} - 1\right]^2 + \left[\left(\frac{P_F(p^1, p^2, q^1, q^2) p_n^1}{p_n^2}\right) - 1\right]^2\right\}$$

As was the case with the dissimilarity index defined by (5.19), the index defined by (5.20) will equal plus infinity if one of the prices for commodity group n, p_n^1 or p_n^2, equals zero.[39] Again, it is useful to define an imputed price for the zero price to insert into the formula, and a reasonable convention is to use the same imputed prices that were suggested for (5.19)—that is, if $p_n^1 = 0$, then define $p_n^{1*} \equiv \frac{p_n^2}{P_F(p^1, p^2, q^1, q^2)}$, and if $p_n^2 = 0$, then define $p_n^{2*} \equiv p_n^1 P_F(p^1, p^2, q^1, q^2)$.

Recently, Rao, Shankar, and Hajarghasht (2010) used the MST method for constructing PPPs across the member countries of the Organisation for Economic Co-operation and Development (OECD) based on data for 1996. Relying on the PLS and WAQ dissimilarity measures defined by (5.18) and (5.20), they compared the resulting spatial chains with the standard GEKS method. They found some fairly significant differences among the three sets of parities for the 24 countries in the comparison, with differences in the PPP for a single country of up to 10 percent. Thus the choice of method does matter, even if the methods of comparison are restricted to multilateral methods that allow for substitution effects. An interesting aspect of their study is that they found that when the WAQ was used as the dissimilarity measure as opposed to the PLS, the linking of the countries was much more intuitive:

> As is generally the case with MSTs, there are a number of counter intuitive paths. For example, Spain and Greece are connected through Portugal, Denmark, USA, UK, Germany, Switzerland, Austria, Sweden, Italy. Similarly Australia and New Zealand are connected through the UK, Germany, Switzerland and Austria. Now we turn to Figure 2 where MST based on relative price distance measure is provided. The links in WRPD based MST are a lot more intuitive and are consistent with the notion of price similarity of the countries. For example, Spain, Italy, Portugal, Greece and Turkey are all connected directly, USA-Canada has a direct link so is the pair Ireland-United Kingdom. Countries like Sweden, Finland, Iceland, Norway and Denmark are all connected together. The main conclusion emerging from Figure 2 is that the WRPD [WAQ] is a better measure of price similarity than the PLS used in the standard MST applications. (Rao, Shankar, and Hajarghasht 2010)

Thus it appears that the pattern of bilateral links that emerges when using the MST method is much more "sensible" when a more discriminating measure of dissimilarity is used in the linking algorithm, as compared with use of the Paasche-Laspeyres spread measure defined by (5.18). Thus in future applications of the MST method it is recommended that (5.18) not be used as the dissimilarity measure that is a key input for the MST method.

The narrowing of Paasche-Laspeyres spreads by the use of a spatial chaining method is not the only advantage of this method of linking countries; there are also advantages at lower levels of aggregation. If countries similar in structure are compared, generally it will be found that product overlaps are maximized, and therefore the basic heading PPPs will be more accurately determined for countries similar in structure:

> Many differences in quality and proportion of high tech items … are likely to be more pronounced between countries with very different economic structures. If criteria can be developed to identify countries with similar economic structure and they are compared only with each other, then it may overcome many of the issues of quality and lowest common denominator item comparisons. Economically similar countries are likely to have outlet types in similar proportions carrying the same types of goods and services. So direct comparisons between such countries will do a better job of holding constant the quality of the items than comparisons across more diverse countries. (Aten and Heston 2009, 251)

> Using the same spanning tree for a number of years would dramatically simplify multilateral international comparisons. Each country would only have to compare itself with its immediate neighbors in the spanning tree, thus reducing the cost and increasing the

> timeliness of international comparisons. Furthermore, by construction, each country's immediate neighbors in the minimum spanning tree will tend to have similar consumption patterns. This may substantially increase the characteristicity of the comparisons. Geary-Khamis, by contrast, compares all countries using a single average price vector. In a comparison over rich and poor countries the average price vector may bear little resemblance to the actual price vectors of many of the countries in the comparison. Conversely, EKS uses all possible combinations of bilateral comparisons. This also requires all countries to provide price and expenditure data on the same set of basic headings, thus reducing the characteristicity of each comparison. (R. J. Hill 2009, 236–37)

Thus the method of spatial linking, if adopted, would involve some changes to country commodity lists. Each country in the minimum spanning tree would be linked to at least one other country, and so for each bilateral link a list of representative commodities pertaining to that link would have to be priced by the two countries in the link. If a country was a local "star" country and linked to several other countries, then the local star country would have to price out a commodity list that pertained to each pair of bilateral links.

Hill has also pointed out that the basic MST methodology could be adapted to impose a priori restrictions on possible links between certain countries: "Suppose for example . . . we do not want India to be linked directly with Hong Kong [SAR, China]. This *exclusion* restriction can be imposed by replacing the PLS between India and Hong Kong [SAR, China], in the K × K PLS matrix, by a large dummy value. . . . Similarly, suppose we want Korea to be linked directly with Japan. This *inclusion* restriction can be imposed by replacing the PLS measure between Korea and Japan with a small dummy value. . . . This ensures that the corresponding edge is selected" (R. J. Hill 2009, 237).

Finally, Hill has noted that not all statistical agencies produce data of the same quality, and that the MST method can be adapted to take this fact into account: "In particular, some countries have better resourced national statistical offices than others. It would make little sense to put a country with an under resourced national statistical office at the center of a regional star even if so specified by the minimum spanning tree" (R. J. Hill 2009, 237).

The MST algorithm can be modified to ensure that countries with under-resourced statistical offices enter the spanning tree with only one bilateral link to the other countries in the comparison.

To sum up, the *advantages* of the MST method for making multilateral comparisons are as follows:

- The MST method, using a superlative index number formula for forming bilateral links, is, like GEKS, consistent with the economic approach to making multilateral comparisons—that is, it takes into account substitution effects.
- The MST method is likely to lead to a more accurate set of parities than those generated by the GEKS method, because the bilateral links between pairs of countries are based on comparisons between countries with the most similar structures of relative prices—that is, the MST method is the spatial counterpart to chained annual indexes in the time series context.
- The influence of countries with under-resourced statistical agencies can be minimized in a simple modification of the basic MST method.

There are also some *disadvantages* to the spatial linking method:

- The method is not as familiar as GEKS and GK, and hence it will be more difficult to build up a constituency for its use.

- When compared with GEKS, the method does have some arbitrary aspects in that (1) different measures of dissimilarity could be used, and there is no universal agreement at this stage as to which measure is the most appropriate one to use; (2) the treatment of zero "prices" and "quantities" in the measures of dissimilarity is not completely straightforward; and (3) the treatment of countries with under-resourced statistical agencies is also not completely straightforward, and, moreover, it may prove difficult to decide exactly which countries are under-resourced.
- The path of bilateral links between countries generated by the method could be unstable—that is, the minimum spanning tree linking the countries could change when moving from one cross-sectional comparison between countries to another cross-sectional comparison.[40]

As of this writing, spatial linking will not be used in the 2011 ICP. Before the MST method is widely adopted, it will be necessary to do more experimentation and trial runs using the method.

5.7 An Artificial Data Set Numerical Example

Diewert (1999, 79–84) illustrated the differences between various multilateral methods by constructing country PPPs and shares of "world" final demand volumes for a three-country, two-commodity example. The GEKS, GK, IDB, and MST parities are calculated in this section using his numerical example.

The price and quantity vectors for the three countries are

$$p^1 \equiv [1, 1]; \quad p^2 \equiv \left[10, \frac{1}{10}\right]; \quad p^3 \equiv \left[\frac{1}{10}, 10\right]; \tag{5.21}$$

$$q^1 \equiv [1, 2]; \quad q^2 \equiv [1, 100]; \quad q^3 \equiv [1000, 10].$$

Note that the geometric average of the prices in each country is one, so that average price levels are roughly comparable across countries, except that in country 2 the price of commodity 1 is very high and the price of commodity 2 is very low, and vice versa for country 3. As a result of these price differences, in country 2 consumption of commodity 1 is relatively low and consumption of commodity 2 is relatively high, and vice versa in country 3. Country 1 can be regarded as a tiny country, with total expenditure (in national currency units) equal to three; country 2 is a medium country with total expenditure equal to 20; and country 3 is a large country with expenditure equal to 200.

The Fisher (1922) quantity index Q_F can be used to calculate the volume Q^k of each country k relative to country 1—that is, calculate $\frac{Q^k}{Q^1}$ as $Q_F(p^1, p^k, q^1, q^k) \equiv \left[\frac{p^1 \cdot q^k p^k \cdot q^k}{p^1 \cdot q^1 p^k \cdot q^1}\right]^{\frac{1}{2}}$ for $k = 2, 3$. Set Q^1 equal to 1.0, thereby determining Q^2 and Q^3. These volumes using country 1 as the base or star country are reported in the Fisher 1 column of table 5.1. In a similar manner, taking country 2 as the base, use the Fisher formula to calculate Q^1, $Q^2 = 1$, and Q^3. Then divide these numbers by Q^1, thereby obtaining the numbers listed in the Fisher 2 column of table 5.1. Finally, taking country 3 as the base, use the Fisher formula to calculate Q^1, Q^2, and $Q^3 = 1$. Then divide these numbers by Q^1 and obtain the numbers listed in the Fisher 3 column of table 5.1. Ideally, these

TABLE 5.1 Fisher Star, GEKS, GK, and IDB Relative Volumes for Three Countries

	Fisher 1	Fisher 2	Fisher 3	GEKS	GK	IDB
Q^1	1.00	1.00	1.00	1.00	1.00	1.00
Q^2	8.12	8.12	5.79	7.26	47.42	33.67
Q^3	57.88	81.25	57.88	64.81	57.35	336.67

Note: GEKS = Gini-Éltetö-Köves-Szulc; GK = Geary-Khamis; IDB = Iklé-Dikhanov-Balk.

Fisher star parities would all coincide, but because they do not, take their geometric mean and obtain the GEKS parities listed in the fourth column of table 5.1. Thus for this example, the GEKS economic approach to forming multilateral quantity indexes leads to the volumes of countries 2 and 3 being equal to 7.26 and 64.81 times the volume of country 1.[41]

Turning to the spatial linking method, one can see that country 1 has the price structure most similar to those of both countries 2 and 3—that is, countries 2 and 3 have the most dissimilar structure of relative prices.[42] Thus in this case, the spatial linking method leads to the Fisher star parities for country 1—that is, the spatial linking relative outputs are given by the Fisher 1 column in table 5.1. Note that these parities are reasonably close to the GEKS parities.

The GK parities for P^k and π_n can be obtained by iterating between equations (5.7) and (5.8) until convergence has been achieved.[43] Once these parities have been determined, Q^k can be determined using equations (5.10). These country volumes are then normalized so that $Q^1 = 1$. The resulting parities are listed in the GK column in table 5.1. The GK parity for $\frac{Q^3}{Q^1}$, 57.35, is reasonable, but the parity for $\frac{Q^2}{Q^1}$, 47.42, is much too large to be reasonable from an economic perspective. The cause of this unreasonable estimate for Q^2 is the fact that the GK international price vector, $[\pi_1, \pi_2]$, is equal to [1, 9.00] so that these relative prices are closest to the structure of relative prices in country 3, the large country. Thus the relatively large consumption of commodity 2 in country 2 receives an unduly high price weight using the GK vector of international reference prices, leading to an exaggerated estimate for its volume, Q^2. This illustrates a frequent criticism of the GK method: the structure of international prices to which it gives rise is "biased" toward the price structure of the biggest countries.

The IDB parities for this numerical example are now calculated to determine whether the method can avoid the unreasonable results generated by the GK method. The parities for P^k and π_n can be obtained by iterating between equations (5.13) and (5.14) until convergence has been achieved.[44] Once these parities have been determined, the Q^k can be determined using equations (5.10). These country volumes are then normalized so that $Q^1 = 1$. The resulting parities are listed in the IBD column in table 5.1. The GK parity for $\frac{Q^2}{Q^1}$ is 33.67, which is well outside the suggested reasonable range (from the viewpoint of the economic approach) of 5–9, and the GK parity for $\frac{Q^3}{Q^1}$ is 336.7, which is well outside the suggested reasonable range of 50–90. What is the cause of these problematic parities?

The problematic IDB volume estimates are not caused by an unrepresentative vector of international prices, because the IBD international price vector, $[\pi_1, \pi_2]$, is equal to [1, 1], which in turn is equal to the vector of (equally weighted) geometric mean commodity prices across countries. The problem is that no additive method can take into account the problem of declining marginal utility as consumption increases if three or more countries are in the comparison. Thus the IBD vector of international prices $\pi = [1, 1]$ is exactly equal to the country 1 price vector $p^1 = [1, 1]$, and so the use of these international prices results in an accurate volume measure for country 1. But the structure of the IBD international prices is far different from the prices facing consumers in country 2, where the price vector is $p^2 \equiv \left[10, \frac{1}{10}\right]$. The very low relative price for commodity 2 leads consumers to demand a relatively large amount of this commodity (100 units), and the relatively high price for commodity 1 leads to a relatively low demand for this commodity (1 unit). Thus at international prices, the output of country 2 is πq^2, which is equal to 101, as compared with its nominal output $p^2 q^2$, which is equal to 20. The use of international prices therefore overvalues the output of country 2 relative to country 1 because the international price of commodity 2 is equal to 1, which is very much larger than the actual price of commodity 2 in country 2 (which is $\frac{1}{10}$). Note that $\frac{Q^2}{Q^1}$ is equal to $\frac{\pi q^2}{\pi q^1} = \frac{101}{3} = 33.67$, an estimate that fails to take into account the declining marginal utility of the relatively large consumption of commodity 2 in country 1. A similar problem occurs when the outputs of countries 1 and 3 are compared using international prices, except in this case the use of international prices tremendously overvalues country 3's consumption of commodity 1. The problem of finding international reference prices that are "fair" for two country comparisons can be solved,[45] but the problem cannot be solved in general if three or more countries are being compared, as was seen in section 5.5.

The tentative conclusion at this point is that additive methods for making international price and quantity comparisons in which there are tremendous differences in the structure of prices and quantities across countries are likely to give rather different answers than methods based on economic approaches. For this reason, it is important that the International Comparison Program provide two sets of results—one set based on a multilateral method such as GEKS or MST that allows for substitution effects and another set based on an additive method such as GK or IDB. Users then can decide which set of estimates to use in their empirical work based on whether they need an additive method (with all of its desirable consistency in aggregation properties) or whether they need a method that allows for substitution effects.

5.8 Conclusion

This chapter discusses four multilateral methods for constructing PPPs and relative volumes for countries in a region. Two of the methods are additive: the Geary-Khamis method and the Iklé-Dikhanov-Balk method. Additive methods are preferred by many users because the components of real GDP add up across countries and across commodities when an additive multilateral method is used.

Which additive method is "best"? The axiomatic properties of the IDB and GK systems are very similar, and so it is difficult to discriminate between the two methods based on their axiomatic properties. The main *advantages of the IDB method* are as follows:

- The IDB international prices are not as influenced as the GK international prices by the structure of relative prices in the biggest countries in the region—that is, the IDB method is more "democratic" than the GK method in its choice of international prices.
- From evidence presented by Deaton and Heston (2010) using the 2005 ICP database, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method may have the advantage that it is an additive method that does not

depart too far from the parities generated by the GEKS method.[46]

The main *advantages of the GK system* are as follows:

- The GK system has been used widely in previous ICP rounds, and so users are familiar with the method and may want to continue to use the results of this method.
- The GK system is similar in some ways to the construction of national accounts data when quantities are aggregated over regions, and thus GK estimates may be regarded as a reasonable extension of countrywide national accounts to the world.

The other two methods discussed in this chapter are the GEKS method and the minimum spanning tree method of similarity or spatial linking developed by R. J. Hill using Fisher ideal indexes as basic bilateral building blocks. These two methods can be regarded as being consistent with an economic approach to a multilateral method—that is, these methods deal adequately with substitution behavior on the part of the purchasers of a country's outputs. The spatial linking method was not used in the 2005 ICP, but it has some attractive features, which were discussed in section 5.6.

ANNEX

The Properties of the Iklé-Dikhanov-Balk Multilateral System

Multilateral index number theory is much more complicated than bilateral index number theory. Thus a rather long annex is required to investigate the axiomatic and economic properties of the IDB multilateral system, particularly when some prices and quantities are allowed to be zero.[47]

There are many equivalent ways of expressing the equations that define the IDB parities. Section 5A.1 lists the alternative systems of equations that can be used to define the method. Section 5A.2 provides proofs of the existence and uniqueness of solutions to the IDB equations. Section 5A.3 considers various special cases of the IDB equations. When there are only two countries so that $K = 2$, a bilateral index number formula is obtained, and this case is considered along with the case in which $N = 2$, so that there are only two commodities. These special cases cast some light on the structure of the general indexes. Finally, section 5A.4 explores the axiomatic properties of the IDB method, and section 5A.5 looks at the system's economic properties.

Throughout this annex it is assumed that the number of countries K and the number of commodities N is equal to or greater than two.

5A.1 Alternative Representations

5A.1.1 The P^k, π_n Representation

The basic data for the multilateral system are the prices and quantities for commodity n in country k at the basic heading levels p_n^k and q_n^k, respectively, for $n = 1, \ldots, N$ and $k = 1, \ldots, K$, where the number of basic heading categories N is greater than or equal to two and the number of countries K is greater than or equal to two. The $N \times 1$ vectors of prices and quantities for country k are denoted by p^k and q^k, and their inner product is $p^k q^k$ for $k = 1, \ldots, K$. The share of country k expenditure on commodity n is denoted by $s_n^k \equiv \dfrac{p_n^k q_n^k}{p^k q^k}$ for $k = 1, \ldots, K$ and $n = 1, \ldots, N$.

It is assumed that for each n and k, either p_n^k, q_n^k, and s_n^k are all zero or p_n^k, q_n^k, and s_n^k are all positive. Thus the possibility that some countries do not consume all of the basic heading commodities is taken into account. This factor complicates the representations of the equations because division by zero prices, quantities, or shares leads to difficulties and complicates proofs of existence.[48] For now, the following assumptions are made:

Assumption 1: For every basic heading commodity n, there exists a country k such that p_n^k, q_n^k, and s_n^k are all positive so that each commodity is demanded by some country.

Assumption 2: For every country k, there exists a commodity n such that p_n^k, q_n^k, and s_n^k are all positive so that each country demands at least one basic heading commodity.

In section 5A.1, these assumptions will be strengthened to ensure that the IDB equations have unique, positive solutions.

Recall that the IDB multilateral system was defined by the Dikhanov equations (5.13) and (5.14), plus one normalization such as (5.9). Taking into account the division-by-zero problem, these equations can be rewritten as[49]

$$\text{(5A.1)} \qquad \pi_n = \frac{\left[\sum_{j=1}^{K} s_n^j\right]}{\left[\sum_{k=1}^{K}\left(\frac{q_n^k P^k}{p^k q^k}\right)\right]}; \quad n = 1, \ldots, N$$

and

$$\text{(5A.2)} \qquad P^k = \frac{p^k q^k}{\pi q^k}; \quad k = 1, \ldots, K$$

where π is a vector whose components are $\pi_1, \ldots, \pi_N$.

Under assumptions 1 and 2, equations (5A.1) and (5A.2) will be well behaved even if some p_n^k and q_n^k are zero. Equations (5A.1) and (5A.2) (plus a normalization on the P^k or π_n such as $P^1 = 1$ or $\pi_1 = 1$) provide the second representation of the IDB multilateral equations.[50]

To find a solution to equations (5A.1) and (5A.2), one can start by assuming that $\pi = 1_N$, a vector of ones, and then use equations (5A.2) to determine a set of P^k. These P^k can then be inserted into equations (5A.1) to determine a new π vector. This new π vector can in turn be inserted into equations (5A.2) to determine a new set of P^k, and so on. The process can be continued until convergence is achieved.

5A.1.2 An Alternative P^k, π_n Representation using Biproportional Matrices

Equations (5A.1) and (5A.2) can be rewritten as

$$\text{(5A.3)} \qquad \sum_{k=1}^{K} q_n^k \left[p^k q^k\right]^{-1} \pi_n P^k = \sum_{j=1}^{K} s_n^j; \quad n = 1, \ldots, N$$

and

$$\text{(5A.4)} \qquad \sum_{n=1}^{N} q_n^k \left[p^k q^k\right]^{-1} \pi_n P^k = \sum_{n=1}^{N} s_n^j = 1; \quad k = 1, \ldots, K.$$

Define the $N \times K$ normalized quantity matrix A, which has element a_{nk} in row n and column k where

$$\text{(5A.5)} \qquad a_{nk} \equiv \frac{q_n^k}{p^k q^k}; \quad n = 1, \ldots, N;\ k = 1, \ldots, K.$$

Define the $N \times K$ expenditure share matrix S, which has the country k expenditure share for commodity n, s_n^k in row n and column k. Let 1_N and 1_K be vectors of ones of dimension N and K, respectively. Then equations (5A.3) and (5A.4) can be written in matrix form as[51]

$$\text{(5A.6)} \qquad \hat{\pi} A P = S 1_K$$

and

$$\text{(5A.7)} \qquad \pi^T A \hat{P} = 1_N^T S$$

where $\pi \equiv [\pi_1, \ldots, \pi_N]$ is the vector of IDB international prices, $P \equiv [P^1, \ldots, P^K]$ is the vector of IDB country PPPs, $\hat{\pi}$ denotes an $N \times N$ diagonal matrix with the elements of the vector π along the main diagonal, and $\hat{P}$ denotes an $K \times K$ diagonal matrix with the elements of the vector P along the main diagonal. There are N equations in (5A.6), and K equations in (5A.7). However, examination of (5A.6) and (5A.7) reveals that if $N + K - 1$ of these equations is satisfied, then the remaining equation is also satisfied. Equations (5A.6) and (5A.7) are a special case of the biproportional matrix fitting model from Deming and Stephan (1940) in the statistics context and from Stone (1962) in the economics context (the RAS method). Bacharach (1970, 45) studied this model in great detail and provided rigorous conditions for the existence of a unique positive π, P solution set to (5A.6), (5A.7), and a normalization such as $P^1 = 1$ or $\pi_1 = 1$.[52] In section 5A.1, Bacharach's analysis is used to provide simple sufficient conditions for the existence and uniqueness of a solution to equations (5A.6) and (5A.7) (plus a normalization).

To find a solution to (5A.6) and (5A.7), one can use the procedure suggested at the end of section 5A.1.1, because equations (5A.1) and (5A.2) are equivalent to (5A.3) and (5A.4).[53] Experience with the RAS method has shown that this procedure tends to converge quite rapidly.

5A.1.3 The Q^k, π_n Representation

The previous representations of the IDB system are in terms of a system of equations involving the N international reference prices, π_n, and the K country PPPs, P^k. It is useful to substitute equations (5.8) in the main text. In those equations, $Q^k = \dfrac{p^k q^k}{P^k}$ defines the country volumes or aggregate quantities Q^k in terms of the country k price and quantity vectors (p^k and q^k) and the country k aggregate PPP, (P^k) into equations (5A.1) and (5A.2) in order to obtain the following representation of the IDB multilateral system in terms of the Q^k and the π_n:

$$\text{(5A.8)} \qquad \pi_n = \frac{\left[\sum_{j=1}^{K} s_n^j\right]}{\left[\sum_{k=1}^{K}\left(\frac{q_n^k}{Q^k}\right)\right]}; \quad n = 1, \ldots, N$$

and

$$\text{(5A.9)} \qquad Q^k = \pi q^k; \quad k = 1, \ldots, K.$$

A normalization such as $Q^1 = 1$ or $\pi_1 = 1$ needs to be added to obtain a unique positive solution to (5A.8) and (5A.9).[54] A biproportional iteration process could be set up to find a solution to equations (5A.8) and (5A.9) along the lines suggested at the end of section 5A.1.1, except that now the Q^k are determined rather than the P^k.

5A.1.4 The Q^k Representation

If equations (5A.8) are substituted into equations (5A.9), the following K equations are obtained, involving only the country volumes, $Q^1, \ldots, Q^K$:

$$\text{(5A.10)} \qquad Q^k = \sum_{n=1}^{N}\left\{\frac{\left[s_n^1 + \cdots + s_n^K\right] q_n^k}{\left[\left(\frac{q_n^1}{Q^1}\right) + \cdots + \left(\frac{q_n^K}{Q^K}\right)\right]}\right\}; \quad k = 1, \ldots, K.$$

A normalization such as $Q^1 = 1$ on the Q^k is required to obtain a unique solution. It also can be seen that the K equations (5A.10) are not independent—that is, if both sides of equation k in (5A.10) are divided by Q^k for each k and then the resulting equations are summed, the identity K equals K is obtained, using the fact that $\sum_{n=1}^{N} s_n^k = 1$ for each k. Thus once any $K - 1$ of the K equations in (5A.10) is satisfied, the remaining equation is also satisfied.

Equations (5A.10) can be used in an iterative fashion to obtain a $Q^1, \ldots, Q^K$ solution—that is, make an initial guess at these volume parities and calculate the right-hand side of each equation in (5A.10). This will generate a new set of volume parities that can then be normalized to satisfy, say, $\sum_{k=1}^{K} Q^k = 1$. Then these new volume parities can again be inserted into the right-hand side of equations (5A.10), and so on.[55]

5A.1.5 The P^k Representation

If equation $Q^k = \dfrac{p^k q^k}{P^k}$ is substituted into equations (5A.10), the following K equations involving only the country PPPs, $P^1, \ldots, P^K$, are obtained:

$$(5A.11) \qquad (P^k)^{-1} = \sum_{n=1}^{N} \left\{ \frac{\left[s_n^1 + \cdots + s_n^K\right]\left[\dfrac{q_n^k}{p^k q^k}\right]}{\left[\left(\dfrac{P^1 q_n^1}{p^1 q^1}\right) + \cdots + \left(\dfrac{P^K q_n^K}{p^K q^K}\right)\right]} \right\}; \quad k = 1, \ldots, K.$$

As usual, a normalization such as $P^1 = 1$ on the P^k is needed to obtain a unique solution. The K equations (5A.11) are not independent—that is, if both sides of equation k in (5A.11) are multiplied by P^k for each k and then the resulting equations are summed, the identity K equals K is obtained, using the fact that $\sum_{n=1}^{N} s_n^k = 1$ for each k. Thus once any $K - 1$ of the K equations in (5A.11) are satisfied, the remaining equation is also satisfied.

Equations (5A.11) can be used iteratively to find a solution in a manner similar to the method described at the end of section 5A.1.4.

Equations (5A.10) and (5A.11) are difficult to interpret at this level of generality, but when the axiomatic properties of the method are studied, it will be seen that the IDB parities have good axiomatic properties.

5A.1.6 The π_n Representation

Finally, substitute equations (5A.2) into equations (5A.1) to obtain the following system of N equations that characterize the IDB international prices π_n:

$$(5A.12) \qquad \sum_{k=1}^{K} \left[\frac{\pi_n q_n^k}{\pi q^k}\right] = \sum_{k=1}^{K} s_n^k; \quad n = 1, \ldots, N.$$

Equations (5A.12) are homogeneous of degree zero in the components of the π vector, and so a normalization such as $\pi_1 = 1$ is required to obtain a unique positive solution. If the N equations in (5A.12) are summed, the identity K equals K is obtained, and so if any $N - 1$ of the N equations in (5A.12) are satisfied, then so is the remaining equation.

Equations (5A.12) can be rewritten as

$$\pi_n = \frac{\left[\sum_{k=1}^{K} s_n^k\right]}{\left[\dfrac{\sum_{k=1}^{K} q_n^k}{\pi q^k}\right]}; \quad n = 1; \ldots, N. \tag{5A.13}$$

Equations (5A.13) can be used iteratively in the usual manner to obtain a solution to equations (5A.12).

Equations (5A.12) have an interesting interpretation. Using the international reference prices π_n, define country k's expenditure share for commodity n using these international prices as

$$\sigma_n^k \equiv \frac{\pi_n q_n^k}{\pi q^k}; \quad k = 1, \ldots, K; n = 1, \ldots, N. \tag{5A.14}$$

Substituting (5A.14) into (5A.12) leads to

$$\sum_{k=1}^{K} \sigma_n^k = \sum_{k=1}^{K} s_n^k; \quad n = 1, \ldots, N. \tag{5A.15}$$

Thus for each basic heading commodity group n, the international prices π_n are chosen by the IDB method to be such that the sum over countries' expenditure shares for commodity n using the international reference prices $\sum_{k=1}^{K} \sigma_n^k$ is equal to the corresponding sum over countries' expenditure shares using domestic prices in each country, $\sum_{k=1}^{K} s_n^k$, and this equality holds for all commodity groups n.[56]

5A.2 Conditions for the Existence and Uniqueness of Solutions to the IDB Equations

The biproportional matrix representation explained in section 5A.1.2 is used to find conditions for positive solutions to any set of the IDB equations.[57]

Bacharach (1970, 43–59) provided very weak sufficient conditions for the existence of a strictly positive solution $\pi_1, \ldots, \pi_N, P^1, \ldots, P^K$ to equations (5A.3) and (5A.4), assuming that assumptions 1 and 2 also hold. His conditions involve the concept of matrix connectedness. Let A be an $N \times K$ matrix. Then Bacharach (1970, 44) defines A to be disconnected if after a possible reordering of its rows and columns it can be written in block rectangular form as

$$A = \begin{bmatrix} A_{n\times k} & 0_{n\times(K-k)} \\ 0_{(N-n)\times k} & A_{(N-n)\times(K-k)} \end{bmatrix} \tag{5A.16}$$

where $1 \le n < N$, $1 \le k < K$, $A_{n\times k}$, and $A_{(N-n)\times(K-k)}$ are submatrices of A of dimension $n \times k$ and $N - n \times K - k$, respectively, and $0_{n\times(K-k)}$ and $0_{(N-n)\times(K-k)}$ are $n \times K - k$ and $N - n \times K - k$ matrices of zeros. As Bacharach (1970, 47) noted, the concept of disconnectedness is a generalization to rectangular matrices of the concept of decomposability, which applies to square matrices. Bacharach (1970, 47) defined A to be connected if it is not disconnected (this is a generalization of the concept of indecomposibility, which applies to square matrices). Bacharach (1970, 47–55) went on to show that if the matrix A defined by (5A.5) is connected, assumptions 1 and 2 hold, and if a normalization such as $\pi_1 = 1$ or $P^1 = 1$ is added to equations (5A.3) and (5A.4), then these equations provide a unique positive solution that can be obtained by using the biproportional procedure suggested at the end of section 5A.1.1, which will converge.

It is useful to have somewhat simpler conditions on the matrix A defined by (5A.5), which will imply that it is connected. Either of the following two simple conditions will imply that A is connected (and thus these are sufficient conditions for the existence of unique positive solutions to any representation of the IDB equations):

Condition 1: There exists a commodity n that is demanded by all countries—that is, there exists an n such that $y_n^k > 0$ for $k = 1, \ldots, K$.

Condition 2: There exists a country k that demands all commodities—that is, there exists a k such that $y_n^k > 0$ for $n = 1, \ldots, N$.

Conditions 1 and 2 are easy to check. They will be used in the following section.

5A.3 Special Cases

In this section, some of the general N and K representations of the IDB equations are specialized to cases in which the number of commodities N or the number of countries K is equal to two.

5A.3.1 The Two-Country, Many-Commodity Quantity Index Case

Suppose that the number of countries K is equal to two. Set the country 1 volume equal to one so that Q^1 equals one and the first equation in (5A.10) becomes

$$\text{(5A.17)} \qquad \sum_{n=1}^{N} \left\{ \frac{\left[s_n^1 + s_n^2\right] q_n^1}{\left[q_n^1 + \left(\dfrac{q_n^2}{Q^2}\right)\right]} \right\} = 1.$$

Equation (5A.17) is one equation in the one unknown Q^2, and it implicitly determines Q^2. Q^2 can be interpreted as a Fisher-type bilateral quantity index, $Q_{IDB}(p^1, p^2, q^1, q^2)$, in which p^k and q^k are the price and quantity (or more accurately, volume) vectors for country k. Thus in what follows in the remainder of this section, Q^2 is replaced by Q.

At this point, assume that the data for country 1 satisfy condition 2 (so that q^1, p^1, and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (5A.17). With this condition, the quantity relatives r_n are well defined as

$$\text{(5A.18)} \qquad r_n \equiv \frac{q_n^2}{q_n^1} \geq 0; \quad n = 1, \ldots, N.$$

Assumption 2 implies that at least one quantity relative r_n is positive. Because each q_n^1 is positive and letting Q equal Q^2, (5A.17) can be rewritten using definitions (5A.18) as[58]

$$\text{(5A.19)} \qquad \sum_{n=1}^{N} \left\{ \frac{\left[s_n^1 + s_n^2\right]}{\left[1 + \left(\dfrac{r_n}{Q}\right)\right]} \right\} = 1.$$

Now define the vector of quantity relatives r as $[r_1, \ldots, r_N]$. Then the function on the left-hand side of (5A.19) can be defined as $F(Q, r, s^1, s^2)$, where s^k is the expenditure share vector for country k for $k = 1, 2$. Note that $F(Q, r, s^1, s^2)$ is a continuous, monotonically increasing function of Q for Q positive. It is assumed that the components of q^1 and thus s^1 are all positive. Now compute the limits of $F(Q, r, s^1, s^2)$ as Q tends to plus infinity:

$$\text{(5A.20)} \qquad \lim_{Q\to+\infty} F(Q, r, s^1, s^2) = \sum_{n=1}^{N}\left[s_n^1 + s_n^2\right] = 2.$$

To compute the limit of $F(Q, r, s^1, s^2)$ as Q tends to zero, two cases must be considered. For the first case, assume that both countries consume all commodities so that $q^2 >> 0_N$ (this is in addition to the earlier assumption that $q^1 >> 0_N$). In this case, it is easy to verify that

$$\text{(5A.21)} \qquad \lim_{Q\to 0} F(Q, r, s^1, s^2) = 0.$$

For the second case, assume that one or more components of q^2 are zero, and let N^* be the set of indexes n such that q_n^2 equals zero. In this case,

$$\text{(5A.22)} \qquad \lim_{Q\to 0} F(Q, r, s^1, s^2) = \sum_{n\in N^*} s_n^1 < 1$$

where the inequality in (5A.22) follows from the fact that it is assumed that all s_n^1 are positive and the sum of all s_n^1 is one.

The fact that $F(Q, r, s^1, s^2)$ is a continuous, monotonically increasing function of Q along with (5A.20)–(5A.22) implies that a finite positive Q solution to the equation $F(Q, r, s^1, s^2) = 1$ exists and is unique. Denote this solution as

$$\text{(5A.23)} \qquad Q = G(r, s^1, s^2).$$

Now use the implicit function theorem to show that $G(r, s^1, s^2)$ is a continuously differentiable function that is increasing in the components of r. Thus

$$\text{(5A.24)} \qquad \frac{\partial G(r, s^1, s^2)}{\partial r_n} = \frac{\left[s_n^1 + s_n^2\right]\left[1 + \left(\frac{r_n}{Q}\right)\right]^{-2} Q}{\left\{\sum_{i=1}^{N}\left[s_i^1 + s_i^2\right]\left[1 + \left(\frac{r_i}{Q}\right)\right]^{-2} r_i\right\}} > 0; \quad n = 1, \ldots, N$$

where Q satisfies (5A.23). However, the inequalities in (5A.24) do not imply that the IDB bilateral index number formula $Q_{\text{IDB}}(p^1, p^2, q^1, q^2)$ is increasing in the components of q^2 and decreasing in the components of q^1. The derivatives in (5A.24) were calculated under the hypothesis that r_n equal to $\frac{q_n^2}{q_n^1}$ increased, but the share vectors s^1 and s^2 were held constant as r_n was increased. In fact, it is not the case that $Q_{\text{IDB}}(p^1, p^2, q^1, q^2)$ is globally increasing in the components of q^2 and globally decreasing in the components of q^1.[59]

It is clear that $Q_{\text{IDB}}(p^1, p^2, q^1, q^2)$ satisfies the identity test—that is, if $q^1 = q^2$ so that all quantity relatives r_n equal one, then the only Q that satisfies (5A.19) is $Q = 1$. It is also clear that if $q^2 = \lambda q^1$ for $\lambda > 0$, then $Q_{\text{IDB}}(p^1, p^2, q^1, \lambda q^1) = \lambda$.[60]

Define $\alpha \geq 0$ as the minimum over n of the quantity relatives $r_n = \dfrac{q_n^2}{q_n^1}$ and define $\beta > 0$ as the maximum of these quantity relatives. Then using the monotonicity properties of the function $F(Q, r, s^1, s^2)$ defined by the left-hand side of (5A.19), it can be shown that

$$\text{(5A.25)} \qquad \alpha \leq Q_{\mathrm{IDB}}(p^1, p^2, q^1, q^2) \leq \beta$$

with strict inequalities in (5A.25) if the r_n are not all equal. Thus the IDB bilateral quantity index satisfies the usual mean value test for bilateral quantity indexes.[61]

It is possible to develop various approximations of $Q_{\mathrm{IDB}}(p^1, p^2, q^1, q^2)$ that cast some light on the structure of the index. Recall that (5A.19) defined Q_{IDB} in implicit form. This equation can be rewritten as a weighted harmonic mean equal to two:

$$\text{(5A.26)} \qquad \left\{ \sum_{n=1}^{N} w_n \left[1 + \left(\frac{r_n}{Q} \right) \right]^{-1} \right\}^{-1} = 2$$

where the weights w_n in (5A.26) are defined as

$$\text{(5A.27)} \qquad w_n \equiv \left(\frac{1}{2} \right) \left(s_n^1 + s_n^2 \right); \quad n = 1, \ldots, N$$

Now approximate the weighted harmonic mean on the left–hand side of (5A.26) by the corresponding weighted arithmetic mean to obtain the following approximate version of equation (5A.26):

$$\text{(5A.28)} \qquad \sum_{n=1}^{N} w_n \left[1 + \left(\frac{r_n}{Q} \right) \right] \approx 2$$

Using the fact that the weights w_n sum up to one, (5A.28) implies that $Q = Q_{\mathrm{IDB}}$ is approximately equal to the following expression:

$$\text{(5A.29)} \qquad Q_{\mathrm{IDB}}(r, w) \approx \sum_{n=1}^{N} w_n r_n = \sum_{n=1}^{N} \left(\frac{1}{2} \right) \left[\left(\frac{p_n^1 q_n^1}{p^1 q^1} \right) + \left(\frac{p_n^2 q_n^2}{p^2 q^2} \right) \right] \left[\frac{q_n^2}{q_n^1} \right].$$

If the weighted arithmetic mean on the right-hand side of (5A.29) is further approximated by the corresponding weighted geometric mean, then $Q_{\mathrm{IDB}}(r, w)$ is approximately equal to

$$\text{(5A.30)} \qquad Q_{\mathrm{IDB}}(r, w) \approx \prod_{n=1}^{N} r_n^{w_n} \equiv Q_T(r, w)$$

where Q_T is the logarithm of the Törnqvist Theil quantity index defined as $\ln Q_T = \sum_{n=1}^{N} w_n \ln r_n$. If all of the quantity relatives r_n are equal to the same positive number, say λ, then the approximations in (5A.28)–(5A.30) will be exact, and under these conditions, where q^2 is equal to λq^1, then the following equalities will hold:

$$\text{(5A.31)} \qquad Q_{\mathrm{IDB}}(\lambda 1_N, w) = Q_T(\lambda 1_N, w) = \lambda.$$

In the more general case, where the quantity relatives r_n are about equal to the same positive number so that q^2 is approximately proportional to q^1, then the Törnqvist Theil quantity index

$Q_T(r, w)$ will provide a good approximation of the implicitly defined IDB quantity index, $Q_{IDB}(r; w)$.[62] However, in the international comparison context it is frequently the case that quantity vectors are far from proportional, and in this nonproportional case Q_{IDB} can be rather far from Q_T and other superlative indexes such as Q_F as was seen earlier in section 5.7 of the main text.

5A.3.2 The Two-Country, Many-Commodity Price Index Case

Again, suppose that the number of countries K is equal to two. Set the country 1 PPP, P^1, equal to one and the first equation in (5A.11) becomes

$$\text{(5A.32)} \qquad \sum_{n=1}^{N} \left\{ \frac{(s_n^1 + s_n^2)\left(\dfrac{q_n^1}{p^1 q^1}\right)}{\left[\left(\dfrac{q_n^1}{p^1 q^1}\right) + \left(\dfrac{P^2 q_n^2}{p^2 q^2}\right)\right]} \right\} = 1.$$

Equation (5A.32) is one equation in the one unknown P^2 (the country 2 PPP), and it implicitly determines P^2. P^2 can be interpreted as a Fisher-type bilateral price index, $P_{IDB}(p^1, p^2, q^1, q^2)$, where p^k and q^k are the price and quantity vectors for country k. Thus in what follows, P^2 will be replaced by P.

Again, it is assumed that the data for country 1 satisfy condition 2 (so that p^1, q^1, and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (5A.32). It is convenient to define the country k normalized quantity vector u^k as the country k quantity vector divided by the value of its output in domestic currency, $p^k q^k$:

$$\text{(5A.33)} \qquad u^k \equiv \frac{q^k}{p^k q^k}; \quad k = 1, 2.$$

Because q^1 is strictly positive, so is u^1. Hence definitions (5A.33) can be substituted into (5A.32) to obtain the following equation, which implicitly determines $P^2 = P = P_{IDB}$:

$$\text{(5A.34)} \qquad \sum_{n=1}^{N} \left\{ \frac{[s_n^1 + s_n^2]}{\left[1 + P\left(\dfrac{q_n^2}{q_n^1}\right)\left(\dfrac{p^1 q^1}{p^2 q^2}\right)\right]} \right\} = \sum_{n=1}^{N} \left\{ \frac{[s_n^1 + s_n^2]}{\left[1 + P\left(\dfrac{u_n^2}{u_n^1}\right)\right]} \right\}.$$

Define $r_n \equiv \dfrac{u_n^2}{u_n^1}$ for $n = 1, \ldots, N$ and rewrite P as $\dfrac{1}{Q}$. Equation (5A.34) then becomes equation (5A.19) in the previous section, and so the analysis surrounding equations (5A.19)–(5A.25) can be repeated to give the existence of a positive solution $P(r, s^1, s^2)$ to (5A.34), along with some of the properties of the solution.

Equation (5A.34) can be used to show that the IDB bilateral price index P, which is the solution to (5A.34), regarded as a function of the price and quantity data pertaining to the two countries, $P_{IDB}(p^1, p^2, q^1, q^2)$, satisfies the first 11 of the 13 bilateral tests listed in Diewert (1999, 36).[63] It fails only the monotonicity in the components of p^1 and p^2 tests—that is, it is not necessarily the case that $P_{IDB}(p^1, p^2, q^1, q^2)$ is decreasing in the components of p^1 and increasing in the components of p^2. Thus the axiomatic properties of the IDB bilateral price index are rather good.

The bounds on the IDB bilateral quantity index given by (5A.25) do not have exactly analogous price counterparts. To develop counterparts to the bounds (5A.25), it is convenient to assume that all of the price and quantity data pertaining to both countries are positive. Under these conditions, the N implicit partial price indexes ρ_n can be defined as

$$\text{(5A.35)} \qquad \rho_n \equiv \frac{\left[\dfrac{p^2q^2}{q_n^2}\right]}{\left[\dfrac{p^1q^1}{q_n^1}\right]} = \frac{\left[\dfrac{p^2q^2}{p^1q^1}\right]}{\left[\dfrac{q_n^2}{q_n^1}\right]}; \quad n = 1, \ldots, N.$$

An implicit bilateral price index is defined as the value ratio, $\frac{p^2q^2}{p^1q^1}$, divided by a quantity index, say $Q(p^1, p^2, q^1, q^2)$, where Q is generally some type of weighted average of the individual quantity relatives, $\frac{q_n^2}{q_n^1}$. Thus each quantity relative, $\frac{q_n^2}{q_n^1}$, can be regarded as a partial quantity index, and hence the corresponding implicit quantity index, which is the value ratio divided by the quantity relative, can be regarded as an implicit partial price index. Substitution of definitions (5A.35) into (5A.34) leads to the following equation, which implicitly determines P equal to $P_{\text{IDB}}(p^1, p^2, q^1, q^2)$:

$$\text{(5A.36)} \qquad \sum_{n=1}^{N} \left\{ \frac{\left[s_n^1 + s_n^2\right]}{\left[1 + \left(\dfrac{P}{\rho_n}\right)\right]} \right\} = 1.$$

Define α as the minimum over n of the partial price indexes ρ_n, and define β as the maximum of these partial price indexes. The monotonicity properties of the function defined by the left-hand side of (5A.36) can then be used to establish the following inequalities:

$$\text{(5A.37)} \qquad \alpha \le P_{\text{IDB}}(p^1, p^2, q^1, q^2) \le \beta$$

with strict inequalities in (5A.37) if the ρ_n are not all equal.

An approximate explicit formula for P_{IDB} can be readily developed. Recall that (5A.36) defined P_{IDB} in implicit form. This equation can be rewritten as a weighted harmonic mean equal to two:

$$\text{(5A.38)} \qquad \left\{ \sum_{n=1}^{N} w_n \left[1 + \left(\frac{P}{\rho_n}\right)\right]^{-1} \right\}^{-1} = 2$$

where the weights w_n in (5A.37) are the average expenditure shares, $\left(\frac{1}{2}\right)[s_n^1 + s_n^2]$ for $n = 1, \ldots, N$. Now approximate the weighted harmonic mean on the left-hand side of (5A.37) by the corresponding weighted arithmetic mean to obtain the approximate version of equation (5A.26):

$$\text{(5A.39)} \qquad \sum_{n=1}^{N} w_n \left[1 + \left(\frac{P}{\rho_n}\right)\right] \approx 2.$$

In view of the fact that the weights w_n sum up to one, (5A.39) implies that $P = P_{\text{IDB}}$ is approximately equal to

$$(5A.40) \qquad P_{\text{IDB}}\,(\rho, w) \approx \left[\sum_{n=1}^{N} w_n(\rho_n)^{-1}\right]^{-1} = \left\{\sum_{n=1}^{N}\left(\frac{1}{2}\right)\left[\left(\frac{p_n^1 q_n^1}{p^1 q^1}\right)+\left(\frac{p_n^2 q_n^2}{p^2 q^2}\right)\right]\left[\frac{q_n^2}{q_n^1}\right]\left[\frac{p^1 q^1}{p^2 q^2}\right]\right\}^{-1}$$

where $\rho \equiv [\rho_1, \ldots, \rho_N]$ and $w \equiv [w_1, \ldots, w_N]$. Thus the IDB bilateral price index P_{IDB} is approximately equal to a weighted harmonic mean of the N partial price indexes ρ_n defined earlier by (5A.35).[64]

5A.3.3 The Many-Country, Two-Commodity Case

Consider the case in which there are K countries but only two commodities so that $N = 2$. Recall that equations (5A.2) and (5A.9) determine the IDB country PPPs, P^k, and the country volumes, Q^k, in terms of the country price and quantity vectors, p^k and q^k, and a vector of international reference prices, $\pi \equiv [\pi_1, \ldots, \pi_N]$. Thus once π is determined, P^k and Q^k can be readily determined. In this section, it is assumed that $N = 2$ so that there are only two commodities and K countries. To ensure the existence of a solution to the IDB equations, it is assumed that commodity 1 is consumed by all countries:

$$(5A.41) \qquad q_1^k > 0; \quad k = 1, \ldots, K.$$

The first international prices will be set equal to one:

$$(5A.42) \qquad \pi_1 = 1.$$

Equations (5A.12) determine π_n, but because N equals two, the second equation in (5A.12) can be dropped. Using the normalization (5A.42), the first equation in (5A.12) becomes

$$(5A.43) \qquad \sum_{k=1}^{K} \frac{q_1^k}{\left[q_1^k + \pi_2 q_2^k\right]} = \sum_{k=1}^{K} s_1^k,$$

which determines the international price for commodity 2, π_2.

Using assumptions (5A.41), the country k commodity relatives R^k (the quantities of commodity 2 relative to 1 in country k) are well defined as

$$(5A.44) \qquad R^k \equiv \frac{q_2^k}{q_1^k} \geq 0; \quad k = 1, \ldots, K.$$

Assumption 1 implies that at least one quantity relative R^k is positive. Because each q_1^k is positive, (5A.43) can be rewritten using definitions (5A.44) as[65]

$$(5A.45) \qquad F(\pi_2, R, s_1) \equiv \sum_{k=1}^{K} \frac{1}{\left[1 + \pi_2 R^k\right]} = \sum_{k=1}^{K} s_1^k \equiv s_1$$

where s_1 is defined to be the sum over countries k of the expenditure share of commodity 1 in country k, s_1^k.[66] Define the vector of country quantity relatives R as $[R^1, \ldots, R^K]$. Then the function on the left-hand side of (5A.45) can be defined as $F(\pi_2, R, s_1)$.[67] Note that $F(\pi_2, R, s_1)$ is a continuous, monotonically decreasing function of π_2 for π_2 positive, because the R^k are nonnegative with at least one R^k positive. Now compute the limits of $F(\pi_2, R, s_1)$ as π_2 tends to zero:

$$\lim_{\pi_2 \to 0} F(\pi_2, R, s_1) = K > \sum_{k=1}^{K} s_1^k = s_1. \tag{5A.46}$$

To compute the limit of $F(\pi_2, R, s_1)$ as π_2 tends to plus infinity, consider two cases. For the first case, assume that all countries consume both commodities so that $R >> 0_K$. Using the definition in (5A.45), the following inequality is obtained:

$$\lim_{\pi_2 \to +\infty} F(\pi_2, R, s_1) = 0 < \sum_{k=1}^{K} s_1^k = s_1. \tag{5A.47}$$

For the second case, assume that one or more components of R are zero, and let K^* be the set of indexes k such that R^k equals zero. In this case, the following limit is obtained:

$$\lim_{\pi_2 \to +\infty} F(\pi_2, R, s_1) = \sum_{k \in K^*} s_n^1 < \sum_{k=1}^{K} s_1^k = s_1. \tag{5A.48}$$

The fact that $F(\pi_2, R, s_1)$ is a continuous, monotonically decreasing function of π_2 along with (5A.46)–(5A.48) implies that a finite positive π_2 solution to equation (5A.45) exists and is unique. Denote this solution as $\pi_2 = G(R, s_1)$. It is straightforward to verify that G is decreasing in the components of R and decreasing in s_1.

Suppose that all country quantity relatives R^k are positive, and define α and β to be the minimum and maximum over k, respectively, of these quantity relatives. Then it is also straightforward to verify that π_2 satisfies the following bounds:[68]

$$\frac{\left[\left(\frac{s_1}{K}\right)^{-1} - 1\right]}{\beta} \leq \pi_2 \leq \frac{\left[\left(\frac{s_1}{K}\right)^{-1} - 1\right]}{\alpha}. \tag{5A.49}$$

Thus if all of country quantity relatives $R^k = \frac{q_2^k}{q_1^k}$ are equal to the same positive number λ, then the bounds in (5A.49) collapse to the common value $\frac{\left[\left(\frac{s_1}{K}\right)^{-1} - 1\right]}{\lambda}$.

In the case in which prices and quantities are positive across all countries (so that all R^k are positive), it is possible to rewrite the basic equation (5A.45) in a more illuminating form as

$$\begin{aligned} \sum_{k=1}^{K} s_1^k &= \sum_{k=1}^{K} \frac{1}{\left[1 + \pi_2 R^k\right]} \\ &= \sum_{k=1}^{K} \left\{ \frac{s_1^k}{\left[s_1^k + \pi_2 s_1^k \left(\frac{y_2^k}{y_1^k}\right)\right]} \right\} \\ &= \sum_{k=1}^{K} \left\{ \frac{s_1^k}{\left[s_1^k + \pi_2 s_2^k \left(\frac{p_2^k}{p_1^k}\right)^{-1}\right]} \right\}. \end{aligned} \tag{5A.50}$$

Equation (5A.50) shows that the π_2 that solves the equation is a function of the K country share vectors, $s^1, \ldots, s^K$ (each of which is of dimension 2), and the vector of K country price relatives, $\left[\frac{p_1^2}{p_1^1}, \ldots, \frac{p_K^2}{p_k^1}\right]$. If all of these country price relatives are equal to a common ratio, say $\lambda > 0$, then the solution to (5A.53) is $\pi_2 = \lambda$. In the case in which all of these country price relatives are positive, let α^* and β^* be the minimum and maximum over k, respectively, of these price relatives. Then it is straightforward to verify that π_2 satisfies the following bounds:

$$\alpha^* \le \pi_2 \le \beta^*. \tag{5A.51}$$

5A.3.4 The Two-Country, Two-Commodity Case

In this section, it is assumed that $K = 2$ (two countries) and that $N = 2$ (two commodities). In this case, it is possible to obtain an explicit formula for the country 2 volume Q^2 relative to the country 1 volume Q^1, which is set equal to one—that is, it is possible to obtain an explicit formula for the IDB bilateral quantity index, $Q^2 = Q = Q_{\text{IDB}}(p^1, p^2, q^1, q^2)$. The starting point for this case is equation (5A.17), which determines Q implicitly. In the case in which N equals two, this equation becomes

$$\left\{\frac{\left[s_1^1 + s_1^2\right]q_1^1}{\left[q_1^1 + \left(\frac{q_1^2}{Q}\right)\right]}\right\} + \left\{\frac{\left[\left(1-s_1^1\right) + \left(1-s_1^2\right)\right]q_2^1}{\left[q_2^1 + \left(\frac{q_2^2}{Q}\right)\right]}\right\} = 1. \tag{5A.52}$$

As usual, it is assumed that the data for country 1 are positive so that $q_1^1 > 0$ and $q_2^1 > 0$. Thus the two quantity relatives, $r_n \equiv \frac{q_2^n}{q_1^n}$ for $n = 1, 2$, are well-defined nonnegative numbers. It is assumed that at least one of the relatives r_1 and r_2 are strictly positive. Substitution of these quantity relatives into (5A.52) leads to the following equation for Q:

$$\left\{\frac{\left[s_1^1 + s_1^2\right]Q}{\left[Q + r_1\right]}\right\} + \left\{\frac{\left[\left(1-s_1^1\right) + \left(1-s_1^2\right)\right]Q}{\left[Q + r_2\right]}\right\} = 1. \tag{5A.53}$$

This equation simplifies into the following quadratic equation:[69]

$$Q^2 + \left[s_1^1 + s_1^2 - 1\right]\left[r_2 - r_1\right]Q - r_1 r_2 = 0. \tag{5A.54}$$

In the case in which both r_1 and r_2 are positive, there is a negative and a positive root for (5A.54). The positive root is the desired bilateral quantity index, and it is equal to

$$Q_{\text{IDB}}(p^1, p^2, q^1, q^2) = -\left(\frac{1}{2}\right)(s_1^1 + s_1^2 - 1)(r_2 - r_1) + \left(\frac{1}{2}\right)\left[(s_1^1 + s_1^2 - 1)^2 (r_2 - r_1)^2 + 4r_1r_2\right]^{\frac{1}{2}}. \tag{5A.55}$$

Now suppose that $r_1 = \frac{q_1^2}{q_1^1} = 0$ so that $q_1^1 > 0$ and $q_1^2 = 0$. Then $s_1^2 = 0$ as well. Using (5A.54),

$$Q = \left[1 - s_1^1\right]r_2 = \left[1 - s_1^1\right]\left[\frac{q_2^2}{q_2^1}\right]. \tag{5A.56}$$

Equation (5A.56) makes sense in the present context. Recall that Q is supposed to reflect the country 2 volume or average quantity relative to country 1. If, as a preliminary estimate of this relative volume, Q is set equal to the single nonzero quantity relative r_2, then this would overestimate the average volume of country 2 relative to 1 because country 2 has a zero amount of commodity 1 while country 1 has the positive amount q_1^1. Thus r_2 is scaled down by multiplying it by one minus country 1's share of commodity 1, s_1^1. The bigger this share, the more the preliminary volume ratio r_2 is downsized.

Now suppose that $r_2 = \frac{y_2^2}{y_2^1} = 0$ so that $q_2^1 > 0$ and $q_2^2 = 0$. Then if $s_1^2 = 1$, and using (5A.54),

$$Q = s_1^1 r_1 = \left[1 - s_2^1\right]\left[\frac{q_1^2}{q_1^1}\right]. \tag{5A.57}$$

Again, equation (5A.57) makes sense in the present context. If Q is set equal to the single nonzero quantity relative r_1, then this would overestimate the average volume of country 2 relative to 1 because country 2 has a zero amount of commodity 2 while country 1 has the positive amount q_2^1. Thus scale down r_1 by multiplying it by one minus country 1's share of commodity 2, s_2^1. The bigger this share, the more the preliminary volume ratio r_1 is downsized.

Two other special cases of (5A.54) are of interest. Consider the cases in which the following conditions hold:

$$r_1 = r_2 \tag{5A.58}$$

and

$$s_1^1 + s_1^2 = 1. \tag{5A.59}$$

If either of these two special cases holds, then Q equals $(r_1 r_2)^{\frac{1}{2}}$, the geometric mean of the two quantity relatives. This first result is not surprising because this result is implied by the earlier N commodity results for two countries—see (5A.25). The second result is more interesting. If (5A.59) holds so that the sum of the two country expenditure shares on commodity 1 is equal to one, then the sum of the two country expenditure shares on commodity 2 is also equal to one—that is, it is also the case that $s_2^1 + s_2^2 = 1$ and the IDB quantity index is equal to the geometric mean of the two quantity relatives, $(r_1 r_2)^{\frac{1}{2}}$.[70]

The next section provides a discussion of the axiomatic or test properties of the IDB multilateral system.

5A.4 Axiomatic Properties of the IDB Multilateral System

Recall section 5.4 in the main text of this chapter where 11 tests or axioms for multilateral systems are listed. The axiomatic properties of the IDB system are summarized in the following result.

PROPOSITION 1: Assume that the country price and quantity data P, Q satisfy assumptions 1 and 2 and at least one of the conditions 1 and 2. Then the IDB multilateral system fails only tests 9 and 10 for the 11 tests listed in section 5.4 of the main text.

Proof: The existence and uniqueness of a solution to any one of the representations of the IDB equations are discussed in section 5A.2. The continuity (and once continuous differentiability) of the IDB share functions $S^k(P, Q)$ in the data follow using the Implicit Function Theorem on the system of equations (5A.6) and (5A.7) (plus a normalization) by adapting the arguments in Bacharach (1970, 67–68). This establishes T1.

The proofs of tests T2 and T4–T8 follow by straightforward substitution into equations (5A.10).

The proof of T3 follows by setting π equal to p and then showing that this choice of π satisfies equations (5A.12). Once π has been determined as p, then the Q^k are determined as πq^k for $k = 1, \ldots, K$, and finally the share functions are determined using (5A.15).

The results in section 5A.3.4 can be used to show that T9, the monotonicity test, fails.

The "democratic" nature of the IDB system (each country's shares are treated equally in forming the reference prices π) leads to a failure of test T10.[71]

The main text showed that the IDB method satisfied T11, the additivity test.

Q.E.D.

5A.5 Economic Properties of the IDB Multilateral System

An economic approach to bilateral index number theory was initiated by Diewert (1976) and generalized to multilateral indexes in Diewert (1999, 20–23). The properties of the IDB system in this economic framework are examined in this section.

The basic assumption in the economic approach to multilateral indexes is that the country k quantity vector q^k is a solution to the following country k utility maximization problem:

$$\text{(5A.60)} \qquad \max_q \left\{ f(q) : p^k q = p^k q^k \right\} = u_k = Q^k$$

where $u^k \equiv f(q^k)$ is the utility level for country k, which can also be interpreted as the country's volume Q^k; $p^k >> 0_N$ is the vector of positive prices for outputs that prevail in country k for $k = 1, \ldots, K$;[72] and f is a linearly homogeneous, increasing concave aggregator function that is assumed to be the same across countries. This aggregator function has a dual unit cost or expenditure function $c(p)$, which is defined as the minimum cost or expenditure required to achieve the unit volume level if purchasers face the positive commodity price vector p.[73] Because purchasers in country k are assumed to face the prices $p^k >> 0_N$, the following equalities hold:

$$\text{(5A.61)} \qquad c(p^k) \equiv \min_q \left\{ p^k q : f(q) \geq 1 \right\} \equiv P^k; \quad k = 1, \ldots, K$$

where P^k is the (unobserved) minimum expenditure required for country k purchasers to achieve unit utility or volume level when the purchasers face prices p^k. P^k can also be interpreted as country k's aggregate PPP. Under assumptions (5A.60) it can be shown[74] that the country k price and quantity vectors, p^k and q^k, satisfy

$$\text{(5A.62)} \qquad p^k q = c(p^k) f(q^k) = P^k u_k = P^k Q^k; \quad k = 1, \ldots, K.$$

To make further progress, it is assumed that either the utility function $f(q)$ is once continuously differentiable with respect to the components of q, or the unit cost function $c(p)$ is once continuously differentiable with respect to the components of p (or both).

In the case in which f is assumed to be differentiable, the first-order necessary conditions for the utility maximization problems in (5A.60), along with the linear homogeneity of f, imply the following relationships between the country k price and quantity vectors, p^k and q^k, respectively, and the country unit expenditures e_k, defined in (5A.64):[75]

$$p^k = \nabla f(q^k)P^k; \quad k = 1, \ldots, K \tag{5A.63}$$

where $\nabla f(q^k)$ denotes the vector of first-order partial derivatives of f with respect to the components of q evaluated at the country k quantity vector, q^k.

In the case in which $c(p)$ is assumed to be differentiable, then Shephard's Lemma implies the following equations:

$$q^k = \nabla c(p^k)u_k = \nabla c(p^k)Q^k; \quad k = 1, \ldots, K \tag{5A.64}$$

where $u_k = f(q^k) = Q^k$ denotes the utility level for country k, and $\nabla c(p^k)$ denotes the vector of first-order partial derivatives of the unit cost function c with respect to the components of p evaluated at the country k price vector p^k.

If $f(q)$ or $c(p)$ are differentiable, then because both of these functions are assumed to be linearly homogeneous, Euler's Theorem on homogeneous functions implies the following relationships:

$$f(q^k) = \nabla f(q^k)q^k = \sum_{n=1}^{N} \left[\frac{\partial f(q^k)}{\partial q_n}\right] q_n^k; \quad k = 1, \ldots, K \tag{5A.65}$$

and

$$c(p^k) = \nabla c(p^k)p^k = \sum_{n=1}^{N} \left[\frac{\partial c(p^k)}{\partial p_n}\right] p_n^k; \quad k = 1, \ldots, K. \tag{5A.66}$$

Recall that the expenditure share on commodity n for country k was defined as $s_k^n \equiv \dfrac{p_n^k q_n^k}{p^k q^k}$. In the case in which $f(q)$ is differentiable, substitution of (5A.63) and (5A.65) into these shares leads to

$$s_n^k = \frac{q_n^k f_n(q^k)}{f(q^k)}; \quad n = 1, \ldots, N; k = 1, \ldots, K \tag{5A.67}$$

where $f_n(q^k) \equiv \dfrac{\partial f(q^k)}{\partial q_n}$. In the case in which $c(p)$ is differentiable, substitution of (5A.64) and (5A.66) into the expenditure shares s_n^k leads to

$$s_n^k = \frac{p_n^k c_n(p^k)}{c(p^k)}; \quad n = 1, \ldots, N; k = 1, \ldots, K \tag{5A.68}$$

where $c_n(p^k) \equiv \dfrac{\partial c(p^k)}{\partial p_n}$. Now that the preliminaries have been laid out, it is time to attempt to determine what classes of preferences (i.e., differentiable functional forms for f or c) are consistent with the IDB system of equations (5A.10).

Begin by considering the case of a differentiable utility function $f(q)$, which is positive, increasing, linearly homogeneous, and concave for $q >> 0_N$.[76] Let $q^k >> 0_N$, $Q^k = f(q^k)$, for $k = 1, \dots, K$, and substitute these equations and (5A.67) into equations (5A.10). Then f must satisfy the following system of K functional equations:

$$(5A.69) \qquad \sum_{n=1}^{N} \frac{\left\{\left[\frac{q_n^1 f_n(q^1)}{f(q^1)}\right] + \cdots + \left[\frac{q_n^K f_n(q^K)}{f(q^K)}\right]\right\}\left[\frac{q_n^k}{f(q^k)}\right]}{\left\{\left[\frac{q_n^1}{f(q^1)}\right] + \cdots + \left[\frac{q_n^k}{f(q^k)}\right]\right\}} = 1; \quad k = 1, \dots, K.$$

Note that all of the terms in this system of K equations are the same in each equation except the terms $\frac{q_k^n}{f(q^k)}$ in the middle of equation k. Suppose that $f(q)$ is a linear function of q so that

$$(5A.70) \qquad f(q) = f(q_1, \dots, q_N) = a_1 q_1 + \cdots + a_N q_N; \quad a_1 > 0, \dots, a_N > 0.$$

It is straightforward to verify that the linear function $f(q)$ defined by (5A.70) satisfies the maintained hypotheses on f, and it also satisfies the system of functional equations (5A.69). Thus the IDB multilateral system is consistent with linear preferences.

Now consider the case of a differentiable unit cost function $c(p)$, which is positive, increasing, linearly homogeneous, and concave for $p >> 0_N$. Let $p^k >> 0_N$, $P^k = c(p^k)$ for $k = 1, \dots, K$, and substitute these equations and (5A.64) into equations (5A.10). Then c must satisfy the following system of K functional equations:

$$(5A.71) \qquad \sum_{n=1}^{N} \frac{\left\{\left[\frac{p_n^1 c_n(p^1)}{c(p^1)}\right] + \cdots + \left[\frac{p_n^K c_n(p^K)}{c(p^K)}\right]\right\} c_n(p^k)}{\{c_n(p^1) + \cdots + c_n(p^K)\}} = 1; \quad k = 1, \dots, K.$$

Note that all of the terms in the previous system of K equations are the same in each equation except the partial derivative terms $c_n(p^K)$ in the middle of equation k. Now suppose that $c(p)$ is a linear function of p so that

$$(5A.72) \qquad c(p) = c(p_1, \dots, p_N) = b_1 y_1 + \cdots + b_N y_N; \quad b_1 > 0, \dots, b_N > 0.$$

It is straightforward to verify that the linear function $c(p)$ defined by (5A.72) satisfies the maintained hypotheses on c, and it also satisfies the system of functional equations (5A.71). Thus the IDB multilateral system is consistent with Leontief (no substitution) preferences.

These computations show that the IDB multilateral system is consistent with preferences that exhibit perfect substitutability between commodities (the linear utility function case) and with preferences that exhibit no substitution behavior as prices change (the case of Leontief preferences where the unit cost function is linear). It turns out that if the number of countries is three or more, then these are the only (differentiable) preferences that are consistent with the IDB system as is shown by the following result.

PROPOSITION 2: If the number of countries is greater than two, then the linear utility function defined by (5A.73) is the only regular differentiable utility function that is consistent with the IDB equations (5A.69), and the preferences that are dual to the linear unit cost function defined by (5A.72) are the only differentiable dual preferences that are consistent with the IDB equations (5A.71).

PROOF: Let $K \geq 3$ and let $q^k >> 0_N$ for $k = 1, \ldots, K$. Then the first two equations in (5A.69) can be rearranged as

$$f(q^2) - f(q^1) = \sum_{n=1}^{N} \left\{ \left[\frac{q_n^1 f_n(q^1)}{f(q^1)} \right] + \cdots + \left[\frac{q_n^K f_n(q^K)}{f(q^K)} \right] \right\} \left\{ \frac{\left[q_n^2 - q_n^1 \right]}{\left[\frac{q_n^1}{f(q^1)} + \cdots + \frac{q_n^K}{f(q^K)} \right]} \right\}. \quad (5A.73)$$

Fix n and let the components of q^1 and q^2 satisfy the following assumptions:

$$q_n^2 \neq q_n^1; \quad q_i^2 = q_i^1 \text{ for I} \neq n. \quad (5A.74)$$

Now look at equation (5A.73) when assumptions (5A.74) hold. The left-hand side is independent of the components of q^3, and thus the right-hand side of (5A.73) must also be independent of q^3. Using the linear homogeneity of f, this is sufficient to show that $f_n(q^3)$ must be a constant for any $q^3 >> 0_N$—that is, for all $q >> 0_N$, $f_n(q)$ is equal to a constant a_n, which must be positive under our regularity conditions on f. This proof works for $n = 1, \ldots, N$, which completes the proof of the first part of the proposition.

Let $K \geq 3$ and let $p^k >> 0_N$ for $k = 1, \ldots, K$. Then equations (5A.71) can be rewritten as

$$\sum_{n=1}^{N} \rho_n(p^1, \ldots, p^K) c_n(p^K) = 1; \quad k = 1, \ldots, K \quad (5A.75)$$

where the coefficients $\rho_n(p^1, \ldots, p^K)$ in (5A.75) are defined for $n = 1, \ldots, N$ as

$$\rho_n(p^1, \ldots, p^K) \equiv \frac{\left\{ \left[\frac{p_n^1 c_n(y^1)}{c(p^1)} \right] + \cdots + \left[\frac{p_n^K c_n(p^K)}{c(p^K)} \right] \right\}}{\left\{ c_n(p^1) + \cdots + c_n(p^K) \right\}}. \quad (5A.76)$$

The first two equations in (5A.75) can be subtracted from each other to give

$$\sum_{n=1}^{N} \rho_n(p^1, \ldots, p^K) \left[c_n(p^2) - c_n(p^1) \right] = 0. \quad (5A.77)$$

Then define the vector $\rho(p^1, \ldots, p^K) \equiv [\rho_1(p^1, \ldots, p^K), \ldots, \rho_N(p^1, \ldots, p^K)]$. Because $K \geq 3$, the definitions (5A.76) show that the components of p^3 can be varied (holding the remaining price vectors constant) so that N is found to be linearly independent $\rho(p^1, \ldots, p^K)$ vectors. Substitution of these linearly independent vectors into equation (5A.77) implies that

$$\nabla c(p^2) = \nabla c(p^1). \quad (5A.78)$$

Because equations (5A.78) hold for all positive p^1 and p^2, the partial derivatives of $c(p)$ are constant, which completes the proof of the proposition.
Q.E.D.

Thus the IDB multilateral system suffers from the same defect as the GK system.[77] Neither of these additive systems is consistent with an economic approach that allows consumer preferences to be represented by flexible functional forms, whereas the GEKS system is consistent with preferences that are representable by flexible functional forms.[78]

NOTES

1. The author is indebted to Yuri Dikhanov, D. S. Prasada Rao, Sergey Sergeev, and Frederic A. Vogel for their helpful comments.
2. For additional methods, see Rao (1990), Balk (1996; 2009, 232–60), R. J. Hill (1997, 1999a, 1999b, 2001, 2004, 2009), and Diewert (1999).
3. The five geographic ICP regions in 2005 were Africa, Asia-Pacific, Commonwealth of Independent States (CIS), South America, and Western Asia. The Eurostat-OECD members constituted a sixth region.
4. Iklé (1972, 203) proposed the equations for the method in a rather difficult-to-interpret manner and provided a proof for the existence of a solution for the case of two countries. Dikhanov (1994, 6–9) used the much more transparent equations (5.13) and (5.14) that appear later in this chapter. He also explained the advantages of the method over the GK method and illustrated the method with an extensive set of computations. Balk (1996, 207–8) used the Dikhanov equations and provided a proof of the existence of a solution to the system for an arbitrary number of countries. Van Ijzeren (1983, 42) also used Iklé's equations and provided an existence proof for the case of two countries.
5. These methods can also be used to make comparisons between regions, as will be seen in chapter 6.
6. Fisher (1922, 272–74), in his discussion on comparing the price levels of Norway, Arab Republic of Egypt, and Georgia, came close to introducing this method. Kravis, Heston, and Summers (1982, 104–11) used similarity measures to cluster countries into groups and also came close to introducing Hill's spatial linking method.
7. Note that the expenditures e_n^k are drawn from the national accounts of country k in the reference year and refer to total expenditures on commodity category n—that is, these expenditures are not in per capita terms.
8. National income accountants distinguish between a "quantity" and a "volume." A *volume* is an aggregate of a group of actual quantities. Because country expenditures in each of the basic heading categories are aggregates over many commodities, it is appropriate to refer to q_n^k as volumes rather than quantities. The price levels p_n^k that correspond to q_n^k are called basic heading PPPs.
9. Notation: if $x = [x_1, \ldots, x_N]$, an N dimensional row vector, then x^T denotes the transpose of x and is an N dimensional column vector with the same components. Thus p^k is an N dimensional column vector.
10. Notation: $pq \equiv \sum_{n=1}^{N} p_n q_n$ denotes the inner product between the vectors p and q.
11. Define the country k expenditure share on commodity group n as $s_n^k \equiv \dfrac{p_n^k q_n^k}{p^k q^k}$ for $n = 1, \ldots, N$.

    Then the Laspeyres price index between countries j and k can be written in the following expenditure share form: $P_L(p^k, p^j, q^k, q^j) \equiv \dfrac{p^j q^k}{p^k q^k} = \sum_{n=1}^{N} \left(\dfrac{p_n^j q_n^k}{p^k q^k}\right) = \sum_{n=1}^{N} \left(\dfrac{p_n^j}{p_n^k}\right) \dfrac{p_n^k q_n^k}{p^k q^k} = \sum_{n=1}^{N} \left(\dfrac{p_n^j}{p_n^k}\right) s_n^k$, which is a country k share weighted *arithmetic* mean of the price relatives $\dfrac{p_n^j}{p_n^k}$.
12. Define the country j expenditure share on commodity group n as $s_n^j \equiv \dfrac{p_n^j q_n^j}{p^j q^j}$ for $n = 1, \ldots, N$.

    Then the Paasche price index between countries j and k can be written in the following expenditure share form: $P_P(p^k, p^j, q^k, q^j) \equiv \dfrac{p^j q^j}{p^k q^j} = \left[\sum_{n=1}^{N} \left(\dfrac{p_n^k q_n^j}{p^j q^j}\right)\right]^{-1} = \left[\sum_{n=1}^{N} \left(\dfrac{p_n^j}{p_n^k}\right)^{-1} \left(\dfrac{p_n^j q_n^j}{p^j q^j}\right)\right]^{-1} =$

$\left[\sum_{n=1}^{N}\left(\frac{p_n^j}{p_n^k}\right)^{-1} s_n^j\right]^{-1}$, which is a country j share weighted *harmonic* mean of the price relatives $\frac{p_n^j}{p_n^k}$. Using these formulas for the Laspeyres and Paasche price indexes, it can be seen that the Fisher price index can also be written in terms of expenditure shares and price relatives.

13. See Balk (2008, 91–97) for a review of the literature on axiomatic justifications for the Fisher index.
14. See chapters 15, 16, and 17 in the *Consumer Price Index Manual* (International Labour Organization et al. 2004).
15. For several additional ways of expressing the GEKS PPPs and relative volumes, see Balk (1996), Diewert (1999, 34–37) and section 5.5 of this chapter.
16. All of the multilateral methods described in this section can be applied to subaggregates of the 155 basic heading categories—that is, instead of working out aggregate price and volume comparisons across all 155 commodity classifications, one could just choose to include the food categories in the list of N categories and use the multilateral method to compare aggregate food consumption across countries in the region.
17. An additive multilateral system is sometimes said to have the property of matrix consistency.
18. Hill (1997) and Dikhanov (1994, 5) made this point.
19. What makes the IDB system special is the fact that equations (5.16) are equivalent to equations (5.14). Instead of using harmonic means in equations (5.13) and (5.14), one could use more general means, such as means of order r—that is, equations (5.13) could be replaced by $\pi_n = \left[\frac{\sum_{k=1}^{K} s_n^k \left[\frac{p_n^k}{P^k}\right]^r}{\sum_{j=1}^{K} s_n^j}\right]^{\frac{1}{r}}$ and equations (5.14) by $P^k = \left[\sum_{n=1}^{N} s_n^k \left[\frac{p_n^k}{\pi_n}\right]^r\right]^{\frac{1}{r}}$, where $r \neq 0$. But it is only when $r = -1$ that the second set of equations simplifies to equations (5.16), which implies the additivity of the method.
20. Balk's axioms were somewhat different from those proposed by Diewert because Balk also introduced an extra set of country weights into Diewert's axioms. Balk's example will not be followed here because it is difficult to determine precisely what these country weights should be. For the most up-to-date review of the axiomatic approach to multilateral indexes, see Balk (2008, 232–60).
21. Diewert's test for bilateral consistency in aggregation is omitted, because this test depends on choosing a "best" bilateral quantity index, and there may be no consensus on what this "best" functional form is (Diewert 1999, 18). His final axiom involving the consistency of the multilateral system with the economic approach to index number theory is discussed in section 5.5 of this chapter.
22. Balk (1996, 212) also compares the performance of the two methods (along with other multilateral methods) using his axiomatic system.
23. The fact that big countries play a more important role in determining the international prices when test T10 is satisfied is analogous to a property that national prices have to regional prices when a country's national accounts by product are constructed: the national price for a commodity is taken to be the unit value price for that commodity over regions within the country. Thus large regions with large final demands will have a more important role in determining the national price vector than the smaller regions.
24. The pioneers in this approach were Konüs and Byushgens (1926).
25. Diewert (1974, 113) termed such functional forms *flexible.*

26. Diewert (1976, 117) introduced this concept and terminology.
27. According to Diewert (1999, 50), figure 5.1

> illustrates the Gerschenkron effect: in the consumer theory context, countries whose price vectors are far from the 'international' or world average prices used in an additive method will have quantity shares that are biased upward. . . . It can be seen that these biases are simply quantity index counterparts to the usual substitution biases encountered in the theory of the consumer price index. However, the biases will usually be much larger in the multilateral context than in the intertemporal context since relative prices and quantities will be much more variable in the former context. . . . The bottom line . . . is that the quest for an additive multilateral method with good economic properties (i.e., a lack of substitution bias) is a doomed venture: nonlinear preferences and production functions cannot be adequately approximated by linear functions. Put another way, if technology and preferences were always linear, there would be no index number problem and hundreds of papers and monographs on the subject would be superfluous!

28. Methods that rely on the econometric estimation of preferences across countries are probably not suitable for the ICP, because it becomes very difficult to estimate flexible preferences for 155 commodity categories.
29. One limitation of econometric approaches is that they cannot be used (it is not impossible, but it would be very difficult because there would be 12,000 parameters to estimate in this case).
30. Note that if all countries in the multilateral comparison have proportional "price" vectors, then the GEKS relative volume for any two countries j relative to i, $\frac{S^j}{S^i}$, is simply the Fisher ideal quantity index between the two countries, which in turn is equal to $\frac{p^i q^j}{p^i q^i}$ and to $\frac{p^j q^j}{p^j q^i}$, the Paasche and Laspeyres quantity indexes between the two countries. If a vector of international prices π is chosen to be any one of the country price vectors, then $\frac{S^j}{S^i} = \frac{\pi q^j}{\pi q^i} = \frac{Q^j}{Q^i}$. Thus under the hypothesis of price proportionality across countries, the country real expenditure levels, Q^k, are proportional to πq^k, and the GEKS multilateral method can be regarded as an additive method.
31. This linking methodology was developed by R. J. Hill (1999a, 1999b, 2004, 2009).
32. Perhaps more descriptive labels for the MST method for making international comparisons are the *similarity linking method* or the *spatial chaining method.*
33. Deaton (2010, 33–34) noticed the following problem with the GEKS method. Suppose there are two countries, A and B. The expenditure share on commodity 1 is tiny for country A and very big for country B. Also suppose that the price of commodity 1 in country A is very large relative to the price in country B. Then look at the Törnqvist price index between A and B. The overall price level for country A will be blown up by the relatively high price for good 1 in country A relative to country B and by the big expenditure share in country B on commodity 1. Because the Törnqvist price index will generally closely approximate the corresponding Fisher index, one has ended up exaggerating the price level of country A relative to B. This problem can be mitigated by spatial linking of countries that have similar price and quantity structures.

34. For a more complete discussion of dissimilarity indexes and their properties, see Diewert (2009).
35. Kravis, Heston, and Summers (1982, 105) proposed another similarity measure that is related to a weighted correlation coefficient between two country price or PPP vectors. However, their measure is not a "pure" bilateral similarity measure because their weights depend on the data of all countries in the comparison.
36. If a price p_n^k equals zero, then it is assumed that the corresponding quantity is also zero.
37. Diewert (2009) did not deal with the zero price problem, but it is a real problem that needs to be addressed in order to implement his suggested dissimilarity measures for relative price structures using real data. For additional discussion of the difficulties associated with making comparisons across countries in which different commodities are being consumed, see Deaton and Heston (2010) and Diewert (2010).
38. Some additional examples are presented in chapter 6.
39. If both prices are zero, then simply drop the n-th term in the summation on the right-hand side of (5.20).
40. However, this evidence of unstable links comes from the results of the MST method using the Paasche and Laspeyres spread measure of dissimilarity. Based on the recent research of Rao, Shankar, and Hajarghasht (2010), it is likely that this instability will be reduced if a better measure of dissimilarity is used in the MST algorithm, like those defined by (5.19) and (5.20), as opposed to the use of the PLS measure defined by (5.18).
41. Because the Fisher star parities are not all equal, it must be recognized that the GEKS parities are only an approximation of the "truth." Thus it could be expected that an economic approach would lead to a $\frac{Q^2}{Q^1}$ parity in the 5–9 range and to a $\frac{Q^3}{Q^1}$ parity in the 50–90 range. Note, however, that the IDB parities are well outside these ranges, and the GK parity for $\frac{Q^2}{QY^1}$ is also well outside this suggested range.
42. This MTS result is obtained for all three measures of dissimilarity considered in the previous section—see equations (5.18), (5.19), and (5.20).
43. Only five iterations were required for convergence.
44. Because all of the prices and quantities are positive in this example, equations (5.13) and (5.14) in the main text can be used instead of the more robust (to zero entries) equations (5A.3) and (5A.4) in the annex. Eighteen iterations were required for convergence.
45. See Diewert (1996, 246) for examples of superlative indexes that are additive if there are only two countries or observations.
46. However, the second example in chapter 6 indicates that the IDB parities may not always be closer to the GEKS parities than the GK parities.
47. Balk (1996, 207–8) has written the most extensive published discussion of the properties of the IDB system, but he considers only the case of positive prices and quantities for all commodities across all countries. He does not discuss the economic properties of the method.
48. Balk's existence proof assumed that all prices and quantities were strictly positive (Balk 1996, 208).
49. Equations (5A.1) are equivalent to Balk's equations (38a) in the case in which all price p_n^k are positive, and equations (5A.2) are Balk's equations (38b) (Balk 1996, 207).
50. Equations (5.13) and (5.14) provide a first representation in the case in which all prices and quantities are positive.

51. Notation: when examining matrix equations, vectors such as π and P are to be regarded as column vectors, and π^T and P^T denote their row vector transposes.
52. It is obvious that if the positive vectors π and P satisfy (5A.6) and (5A.7), then $\lambda\pi$ and $\lambda^{-1}P$ also satisfy these equations where λ is any positive scalar. Dikhanov (1997, 12–13) also derived conditions for the existence and uniqueness of the solution set using a different approach.
53. Bacharach (1970, 46) calls this method of solution the biproportional process. He establishes conditions for the existence and uniqueness of a solution to the biproportional process—that is, for the convergence of the process (Bacharach 1970, 46–59). The normalization—say $P^1 = 1$ or $\pi_1 = 1$—can be imposed at each iteration of the biproportional process, or it can be imposed at the end of the process when convergence has been achieved.
54. It can be verified that if $N + K - 1$ of equations (5A.8) and (5A.9) are satisfied, then the remaining equation is also satisfied. Equations (5A.10) may be used to establish this result.
55. When this method was tried on the data for the numerical example in Diewert (1999, 79)—see section 7—it was found that convergence was very slow. The iterative methods described in section A.1.1 converged much more quickly.
56. Dividing both sides of (5A.15) by K means that for each commodity group the average (over countries) expenditure share using the IDB international prices is equal to the corresponding average expenditure share using the domestic prices prevailing in each country.
57. Once the existence and uniqueness of a positive solution to any one of the representations of the IDB equations have been established, using assumptions 1 and 2 it is straightforward to show that a unique positive solution to the other representations is also implied.
58. Equation (5A.19) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.
59. This negative monotonicity result also applies to the Törnqvist Theil bilateral index number formula, Q_T—see Diewert (1992, 221). The logarithm of Q_T is defined as $\ln Q_T = \sum_{n=1}^{N} \left(\frac{1}{2}\right) \left[s_n^1 + s_n^2\right] \ln r_n$.
60. It is also clear from (5A.19) that $Q_{IDB}(p^1, p^2, q^1, q^2)$ satisfies the following four homogeneity tests: $Q_{IDB}(p^1, p^2, q^1, \lambda q^2) = \lambda Q_{IDB}(p^1, p^2, q^1, q^2)$, $Q_{IDB}(p^1, p^2, \lambda q^1, q^2) = \lambda^{-1} Q_{IDB}(p^1, p^2, q^1, q^2)$, $Q_{IDB}(\lambda p^1, p^2, q^1, q^2) = Q_{IDB}(p^1, p^2, q^1, q^2)$, and $Q_{IDB}(p^1, \lambda p^2, q^1, q^2) = Q_{IDB}(p^1, p^2, q^1, q^2)$ for all $\lambda > 0$. Equations (5A.17) or (5A.19) can be used to show that $Q_{IDB}(p^1, p^2, q^1, q^2)$ satisfies the first 11 of Diewert's 13 tests for a bilateral quantity index, failing only the monotonicity in the components of the q^1 and q^2 tests (Diewert 1999, 36). Thus the axiomatic properties of the IDB bilateral quantity index are rather good.
61. See Diewert (1992) for the history of these bilateral tests.
62. If $Q_{IDB}(r)$ and $Q_T(r)$ are regarded as functions of the vector of quantity relatives, then it can be shown directly that $Q_{IDB}(r)$ approximates $Q_T(r)$ to the second order around the point $r = 1_N$.
63. The role of prices and quantities must be interchanged—that is, Diewert tests referred to quantity indexes (Diewert 1999, 36), whereas price indexes are now being considered.
64. The expressions involving the reciprocals of ρ_n require that q^2 be strictly positive (in addition to the maintained assumption that y^1 be strictly positive). Equations (5A.32) and (5A.34) require only that y^1 be strictly positive.
65. Equation (5A.19) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.
66. Note that s_1 satisfies the inequalities $0 < s_1 < K$.

67. Thus the π_2 solution to (5A.45) depends only on the vector of country quantity relatives, R, and the sum across countries k of the expenditure shares on commodity 1, s_1^k. Alternatively, π_2 depends on the K dimensional vector R and the sum across countries commodity share vector, $s^1 + \ldots + s^K$, which is a two-dimensional vector in the present context where $N = 2$.
68. It can be verified that $0 < s_1 < K$ so that $\left(\frac{s_1}{K}\right)^{-1} > 1$. The bounds in (5A.49) are positive when $R >> 0_K$. In the case in which $R > 0_K$, the lower bound is still valid, but the upper bound becomes plus infinity.
69. This equation can be utilized to show that $Q_{IDB}(p^1, p^2, q^1, q^2)$ is not necessarily monotonically increasing in the components of q^1 or monotonically decreasing in the components of q^1.
70. Under these conditions, it is also the case that all prices and quantities are positive in the two countries, because it was assumed that y^1 is strictly positive and y^2 is nonnegative and nonzero—that is, $q^1 >> 0_2$ and $q^2 > 0_2$.
71. Diewert (1999, 27) showed that the GK system satisfied all 11 tests except the homogeneity test, T8, and the monotonicity test, T9. The GK system is a "plutocratic" method in which the bigger countries have a greater influence in determining the international price vector π.
72. In this section, it will be assumed that all country prices and quantities are positive, so that $p^k >> 0_N$ and $q^k >> 0_N$ for $k = 1, \ldots, K$.
73. The unit cost function $c(p)$ is an increasing, linearly homogeneous concave function in p for $p >> 0_N$.
74. See Diewert (1974) for material on duality theory and unit cost functions.
75. See Diewert (1999, 21) for more details on the derivation of these equations.
76. The functions f or c are defined to be regular if they satisfy these regularity conditions.
77. Diewert (1999, 27) showed that when K is greater than or equal to three, the GK system is only consistent with a linear or Leontief aggregator function.
78. See Diewert (1999, 46) for descriptions of multilateral methods that have good economic properties—that is, methods that are consistent with maximizing behavior on the part of consumers with preferences represented by flexible functional forms. See Diewert (1976) for the concept of a flexible functional form and the economic approach to index number theory. In addition to the GEKS system, the weighted and unweighted balanced methods of Own Share, MTS, and van Ijzeren (1983) have good economic properties.

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