International Comparison Program (ICP) Technical Advisory Group (TAG)

Fisher versus Törnqvist when Relative Price Changes are Large

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- **Two related papers:**

- Fisher versus Törnqvist when Relative Price Changes are Large (*Multiplier effect of price variations [uncertainties] on the Fisher index*);

and

- Range of possible results for certain classes of superlative price indices.

Introduction

- Improving ICP methodology is an important topic on the ICP Technical Advisory Group (TAG) agenda. This paper finds the basic aggregation method used in ICP (the Fisher-based GEKS) to be quite unstable in the face of price and expenditure volatility (esp. due to measurement errors).
- The paper discusses the sources of GEKS instability, introduces a model to explain the effects of price shocks, and proposes a remedy for the situation, - the Törnqvist-based CCD (Caves, Christensen and Diewert) index. The results of this paper will be relevant to both spatial and temporal indices.

Temporal price indices

We begin with temporal indices to demonstrate extent of the problem.

For temporal analyses, we utilize the 1990-2023 NIPA PCE data from the US BEA website (bea.gov). Three types of indices are considered: (1) multilateral, or GEKS (multilateral Fisher) and CCD (multilateral Törnqvist), (2) direct, or 2023/1990 direct estimates using only two years, and (3) chained indices. For reference, the Paasche and Laspeyres indices (both the regular and geometric, direct and chained) are provided as well. The chained indices, both based on the Törnqvist and Fisher, are a good approximation of the Divisia index and can serve as our approximation of the "true" price index.

We consider two aggregation scopes: (1) over all items, and (2) over all items without technology items. The fast-moving tech items will be our proxies for volatile items present in the ICP (of course, in the temporal setting those items are "volatile" due to the fast pace of the technical change, not due to measurement errors or other issues).

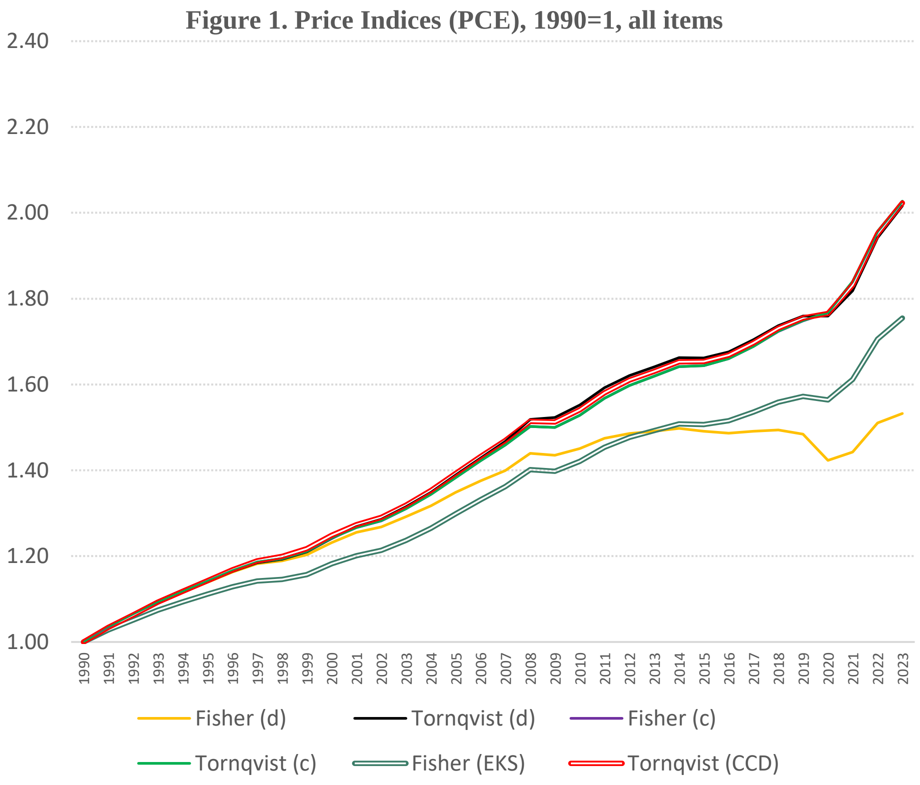
Temporal price indices

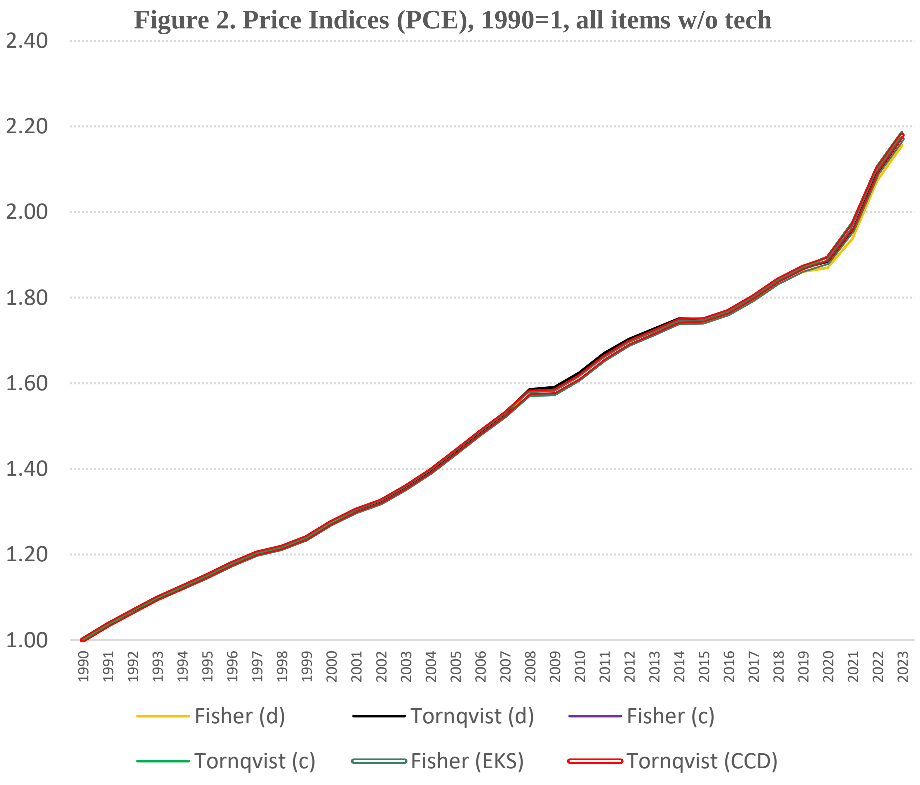
Table A. Price index, 2023 to 1990, NIPA Personal Consumption Expenditures, annual accounts

	2023 indices (1990 = 1.0)		Törnqvist (chained) = 1.0	
	All items [a]	All w/o tech items[1] [b]	All items [c] = [a] / [a.6]	All w/o tech items[1] [d] = [b] / [b.6]
1. Fisher (multilateral), GEKS	1.7547	2.1710	0.8656	0.9927
2. Törnqvist (multilateral), CCD	2.0229	2.1799	0.9979	0.9968
3. Fisher (direct), two-year	1.5324	2.1557	0.7559	0.9857
4. Törnqvist (direct), two-year	2.0162	2.1746	0.9946	0.9943
5. Fisher (chained)	2.0269	2.1872	0.9999	1.0001
6. Törnqvist (chained)	2.0272	2.1869	1.0000	1.0000
reference				
7. Paasche (direct)	1.0117	1.9717	0.4990	0.9016
8. Laspeyres (direct)	2.3213	2.3570	1.1451	1.0777
9. Geometric Paasche (direct)	2.0061	2.2025	0.9896	1.0071
10. Geometric Laspeyres (direct)	2.0264	2.1470	0.9996	0.9817
11. Paasche (chained)	2.0042	2.1738	0.9887	0.9940
12. Laspeyres (chained)	2.0499	2.2006	1.0112	1.0063
13. Geometric Paasche (chained)	2.0566	2.2205	1.0145	1.0153
14. Geometric Laspeyres (chained)	1.9982	2.1539	0.9857	0.9849

Note: Net purchases abroad and NPISH are excluded

Temporal price indices





Temporal price indices

Figure 3. Price Indices (PCE), 1990=1, all items

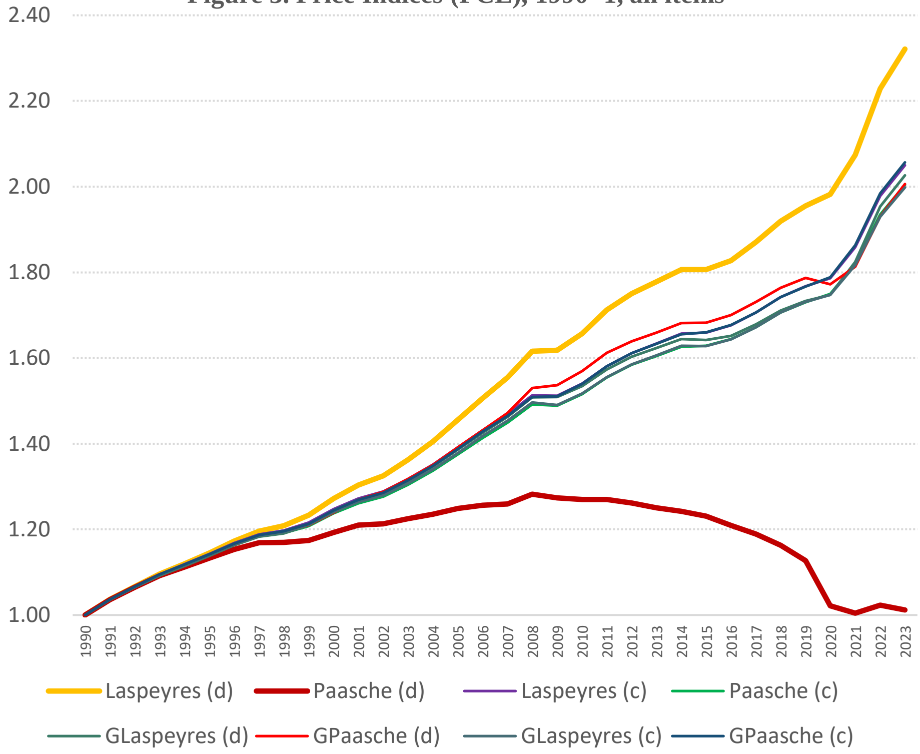
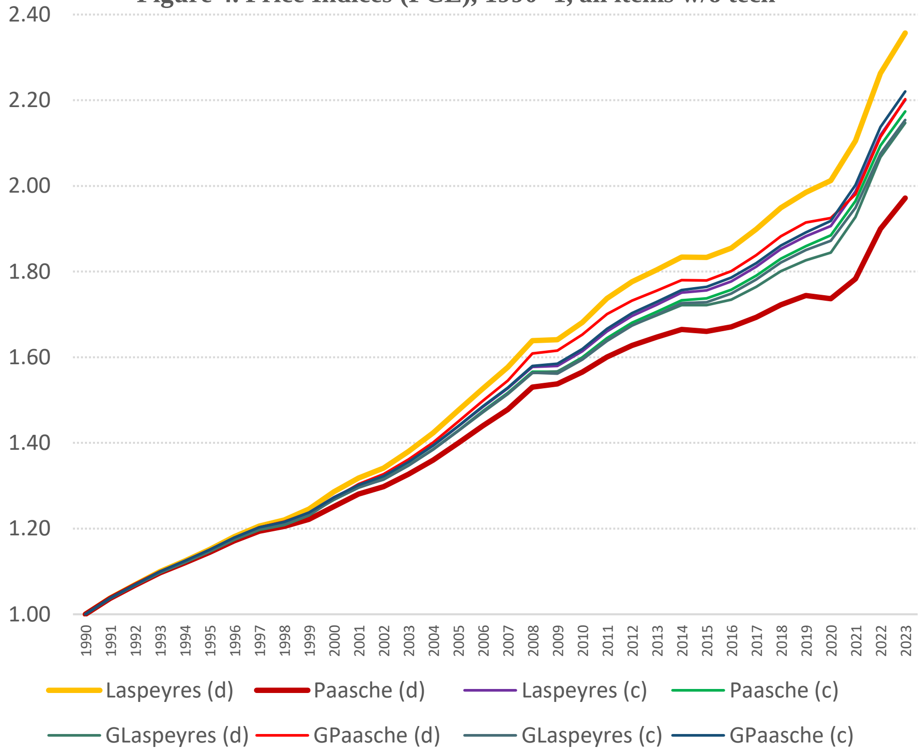
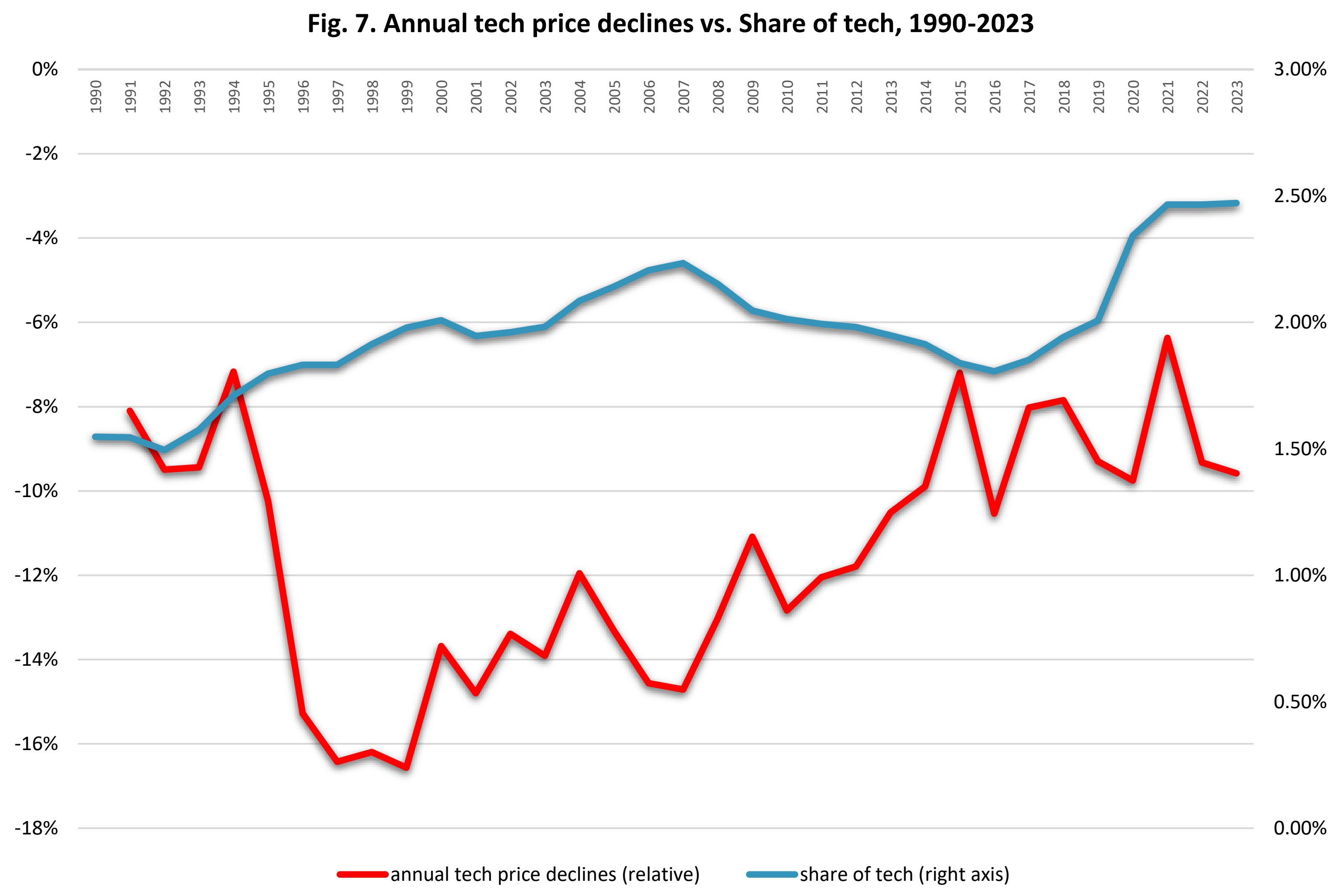


Figure 4. Price Indices (PCE), 1990=1, all items w/o tech



Temporal price indices



Model to simulate effects of price shocks on indices

Consider the following model to simulate price shocks:

1) Expenditure shares are kept constant between the two periods for *all* items.
2) One product [let's call it product 1] with expenditure share *s* experiences a price shock R [i.e., the price of product 1 changes from 1 to R], and prices for all other products are kept constant and equal to one.

We want to compare behavior of the Fisher and Törnqvist indexes as the price of product 1 increases or decreases substantially so that we can ignore other items. We will show that with large movements in relative prices, the Fisher index will show a greater level of volatility than the corresponding Törnqvist index.

This leads to the following expressions for the indices corresponding to our model (Laspeyres, Paasche, and Törnqvist price indices):

$$log\, T(r,s) = log\, e^{\mathrm{s}\log(\mathrm{R})} = s\log(\mathrm{R}) = sr$$

$$log\, La(r,s) = log\,(1 - s + s\,R) = log\,(1 - s + se^{r}\,)$$

$$log\, Pa(r,s) = -log\,(1 - s + s\,/R) = -log\,(1 - s + se^{-r}\,)$$

Model to simulate effects of price shocks on indices

Using Taylor expansions for logarithms and exponentials, and discarding all terms higher than the fourth order, we obtain the following:

$$log\ La(r,s) = sr + \frac{s}{2}(1-s)r^2 + \frac{s}{6}(2s^2 - 3s + 1)r^3 + \frac{s}{24}(1 - 7s + 12s^2 - 6s^3)r^4 + O(r^5)$$

$$log\ Pa(r,s) = sr - \frac{s}{2}(1-s)r^2 + \frac{s}{6}(2s^2 - 3s + 1)r^3 - \frac{s}{24}(1 - 7s + 12s^2 - 6s^3)r^4 + O(r^5)$$

Thus, the Fisher index is approximated as:

$$log\ F(r,s) = sr + s\left(\frac{1}{6} - \frac{s}{2} + \frac{s^2}{3}\right)r^3 + O(r^5)$$

Finally, the distance between the Fisher and Törnqvist indices can be expressed as following:

$$log\ F(r,s) - \ log\ T(r,s) = s\left(\frac{1}{6} - \frac{s}{2} + \frac{s^2}{3}\right)r^3 + O(r^5) \qquad (3a)$$

Model to simulate effects of price shocks on indices

Another derivation:

From Dikhanov (2024a), p.12, [Dikhanov-A-New-Elementary-Index.pdf (unece.org)] we know that in the general case, the log difference between the indices is described by expression (10) of the paper:

$$log\ F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta)\ -\ log\ T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx (\frac{1}{6} - \frac{\beta}{4})\langle \mathbf{d}^3 \rangle \qquad (1)$$

Where $x_i = \frac{p_i^1}{p_i^0} / \prod \left(\frac{p_i^1}{p_i^0}\right)^{\mathbf{s}_{0i}}$, and $\prod \left(\frac{p_i^1}{p_i^0}\right)^{\mathbf{s}_{0i}}$ is the Geometric Laspeyres index (GL). $\langle \mathbf{d}^3 \rangle$, or the third moment, using the notation $d_i = log\ x_i$ and $\langle \mathbf{d}^n \rangle = \sum \mathrm{s}_{0i} {d_i}^n$, is the measure of *skewness* in price changes. It is assumed here that the item shares $\mathbf{s}$ change with prices with elasticity β: $\mathrm{s}_{1i} = \mathrm{s}_{0i} e^{\beta d_i} / \sum \mathrm{s}_{0i} e^{\beta d_i}$.

The expenditure share elasticity β equals zero when all expenditure shares stay the same between the two periods not reacting to price changes, this also means that quantities are inversely proportional to prices; and β equals one when item quantities do not respond to price changes and stay proportional between the two periods.

Setting the share elasticity β to 0 keeps all shares **s** constant between the two periods, and in this case, we can write:

$$log\ F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_0, 0)\ -\ log\ T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_0, 0) \approx \frac{1}{6}\langle \mathbf{d}^3 \rangle$$

Model to simulate effects of price shocks on indices

Noting that log(GL) = sr and r = log(R), with all other prices kept constant, we get

$$\langle \mathbf{d}^3 \rangle = (1-s)(0-sr)^3 + s(r-sr)^3 = s(1-3s+2s^2)r^3$$

From which we immediately obtain:

$$log\,F(r,s) - \ log\,T(r,s) \approx s\left(\frac{1}{6}-\frac{s}{2}+\frac{s^2}{3}\right)r^3 \qquad (2)$$

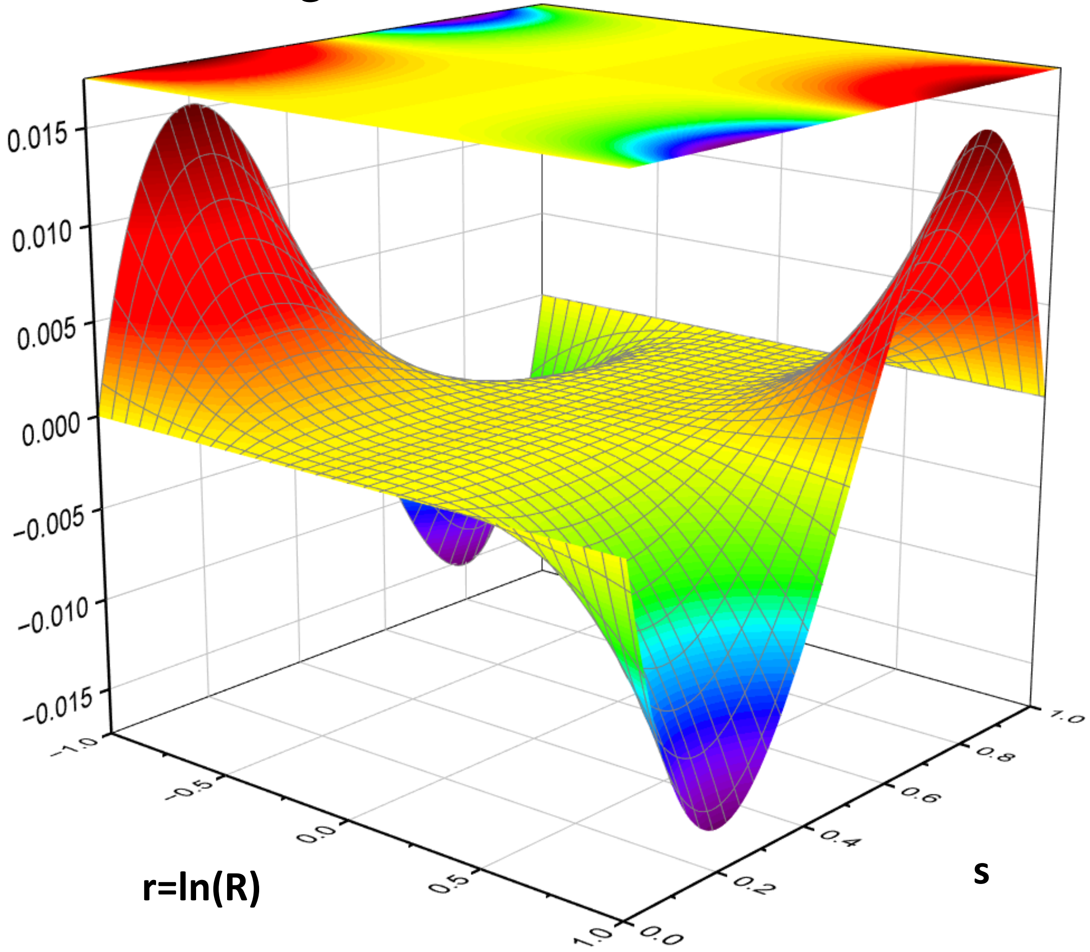
This expression can be re-written as the multiplier effect on the Fisher relative to the Törnqvist as r changes:

$$log\,F(r,s) \approx log\,T(r,s)\left[1+\left(\frac{1}{6}-\frac{s}{2}+\frac{s^2}{3}\right)r^2\right] \qquad (3)$$

The multiplier effect $1+\left(\frac{1}{6}-\frac{s}{2}+\frac{s^2}{3}\right)r^2$ is always greater than one if s < 0.5. Thus, an error in price r would always create a bigger jolt to the Fisher than to the Törnqvist index. This would also work for other price hikes or drops, such as fast-declining tech prices over time. The multiplier effect reaches the maximum at s being around 20% (this is $argmax\left(s(\frac{1}{6}-\frac{s}{2}+\frac{s^2}{3})\right)$, 0≤s≤1, also see Figure A).

Model to simulate effects of price shocks on indices

Figure A. Effect of price shock R on the Törnqvist-Fisher log differences



Explaining the NIPA PCE index discrepancies with the model of the tech items' price shocks

First, it will be useful to investigate higher-level approximations of the model as in our case the price "shocks" are extremely large over the 1990-2023 period. From Dikhanov (2024, *Range of possible results for certain classes of superlative price indices, ANNEX II*) we know that when $\beta = 0$, the distance between the Fisher and Törnqvist indices can be approximated as:

$$log\ F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, 0)\ -\ log\ T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, 0) \approx \frac{1}{6}\langle \mathbf{d}^3 \rangle + \frac{1}{120}(\langle \mathbf{d}^5 \rangle + 10\langle \mathbf{d}^2 \rangle \langle \mathbf{d}^3 \rangle) \qquad (4)$$

Thus, using the same derivations as in the case above for the third-order approximation, the distance between the Törnqvist and Fisher indices using the fifth-order approximation becomes:

$$log\ F(r,s) -\ log\ T(r,s) = s\left(\frac{1}{6} - \frac{s}{2} + \frac{s^2}{3}\right) r^3 + s\left(\frac{1}{120} - \frac{s}{8} + 5\frac{s^2}{12} - \frac{s^3}{2} + \frac{s^4}{5}\right) r^5 + O(r^7) \qquad (5)$$

Or, if **s** is small, the above expression simplifies to:

$$log\ F(r,s) -\ log\ T(r,s) = \frac{s}{6} r^3 + \frac{s}{120} r^5 + O(r^7) \qquad (6)$$

The same results can be obtained by applying Taylor expansions directly to $log\ La(r,s) = log\ (1 - s + se^r\ )$ and $log\ Pa(r,s) = -\ log\ (1 - s + se^{-r}\ )$ as in Section 3.3 of Chapter 3.

Explaining the NIPA PCE index discrepancies with the model of the tech items' price shocks

Let us apply our model to explain the multiplier effect on the GEKS system. The 2023/1990 GEKS index can be written using its constituent binary Fisher indices as:

$$log\ GEKS_{1990}^{2023} = \frac{1}{34}\sum_{1990}^{2023} log\ F_{1990}^{t} + \ log\ F_{t}^{2023} \qquad (7)$$

This expression can be rewritten in the integral form. Thus, assuming that the average rate of relative price declines for the tech products has been constant over the period, and using expression (6) to approximate the Fisher – Törnqvist discrepancies, we obtain the following for $log\ \Delta GEKS_0^T = log\ GEKS_0^T - log\ CCD_0^T$:

$$log\ \Delta GEKS_0^T \approx s/T\int_0^T \left(\frac{1}{6}\left(r\frac{t}{T}\right)^3 + \frac{1}{120}\left(r\frac{t}{T}\right)^5 + \frac{1}{6}\left(r\frac{T-t}{T}\right)^3 + \frac{1}{120}\left(r\frac{T-t}{T}\right)^5\right)dt \qquad (8)$$

Explaining the NIPA PCE index discrepancies with the model of the tech items' price shocks

Here we scaled the time period without loss of generality to [0; T]. Then the tech price shock becomes $r\frac{t}{T}$ for time t, assuming a constant rate of relative price decline for tech items over the period. Taking the definite integral we arrive at (note that T gets canceled out):

$$log\ \Delta GEKS_0^T \approx {}^{s}/_{360}\ r^3(r^2+30) \qquad (9)$$

Now we can find out the ratio of expression (9) to expression (6) under the assumption that s is small:

$$\boldsymbol{log\ \Delta GEKS_0^T\ /(log\ F_0^T - log\ T_0^T) \approx \frac{1}{3} + \frac{10}{3(r^2+20)}} \qquad (10)$$

Accuracy of the simulation

Fig. 10. Fisher-Törnqvist ratio vs. ⟨d^3 ⟩, NIPA 1990-2023, all items

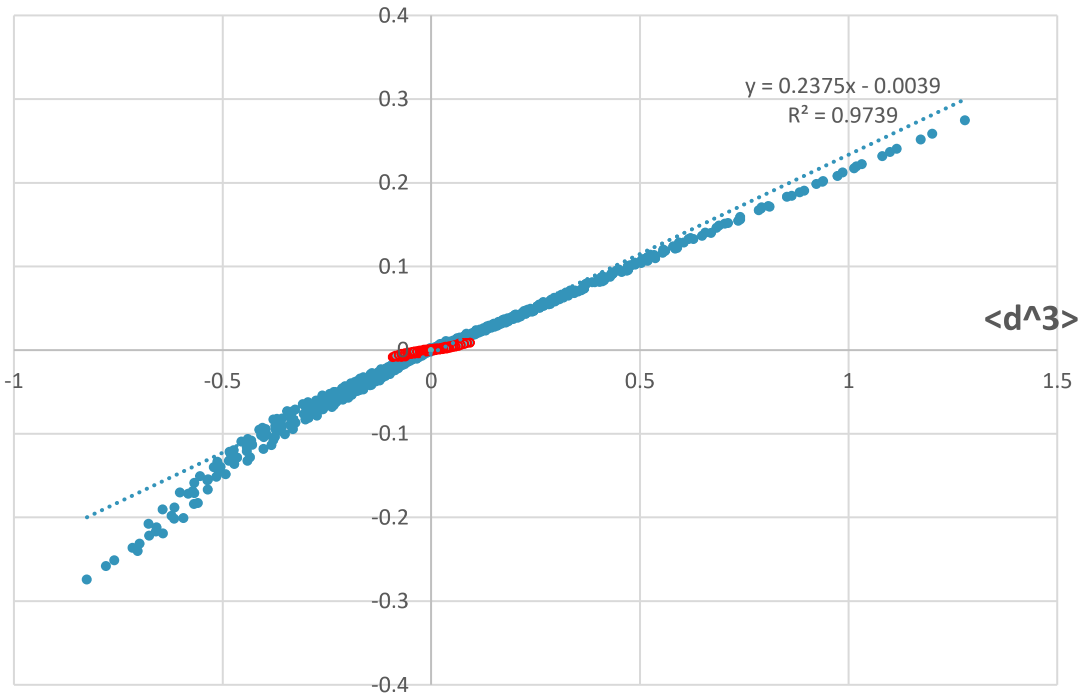
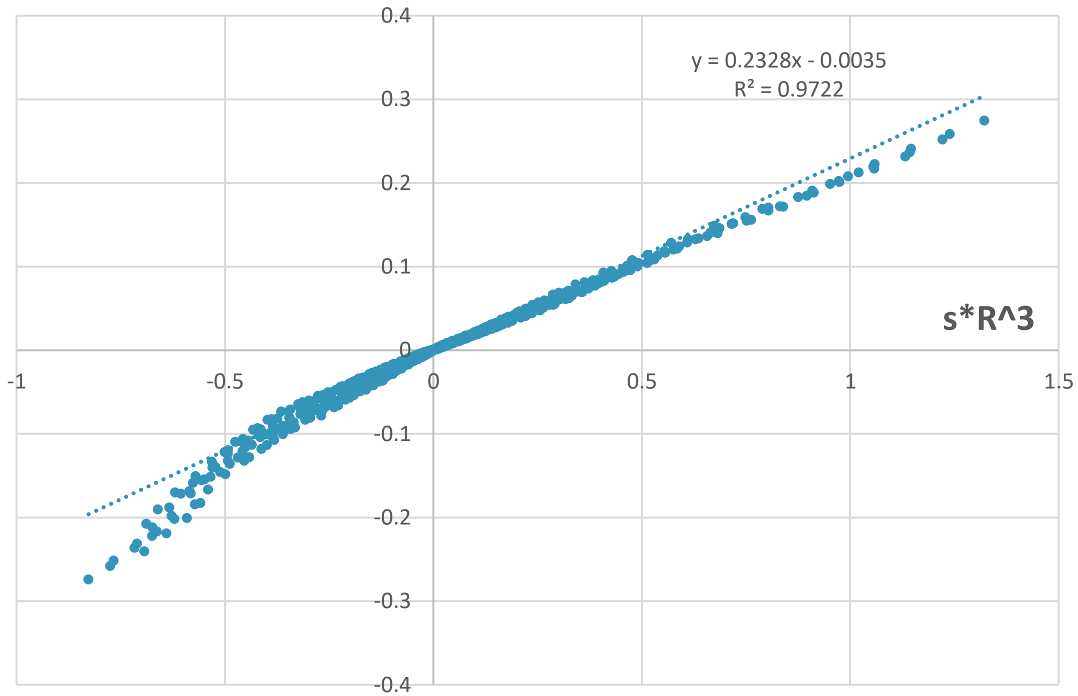


Fig. 11. Fisher-Törnqvist ratio vs. s*R^3 (shocks), NIPA 1990-2023, all items



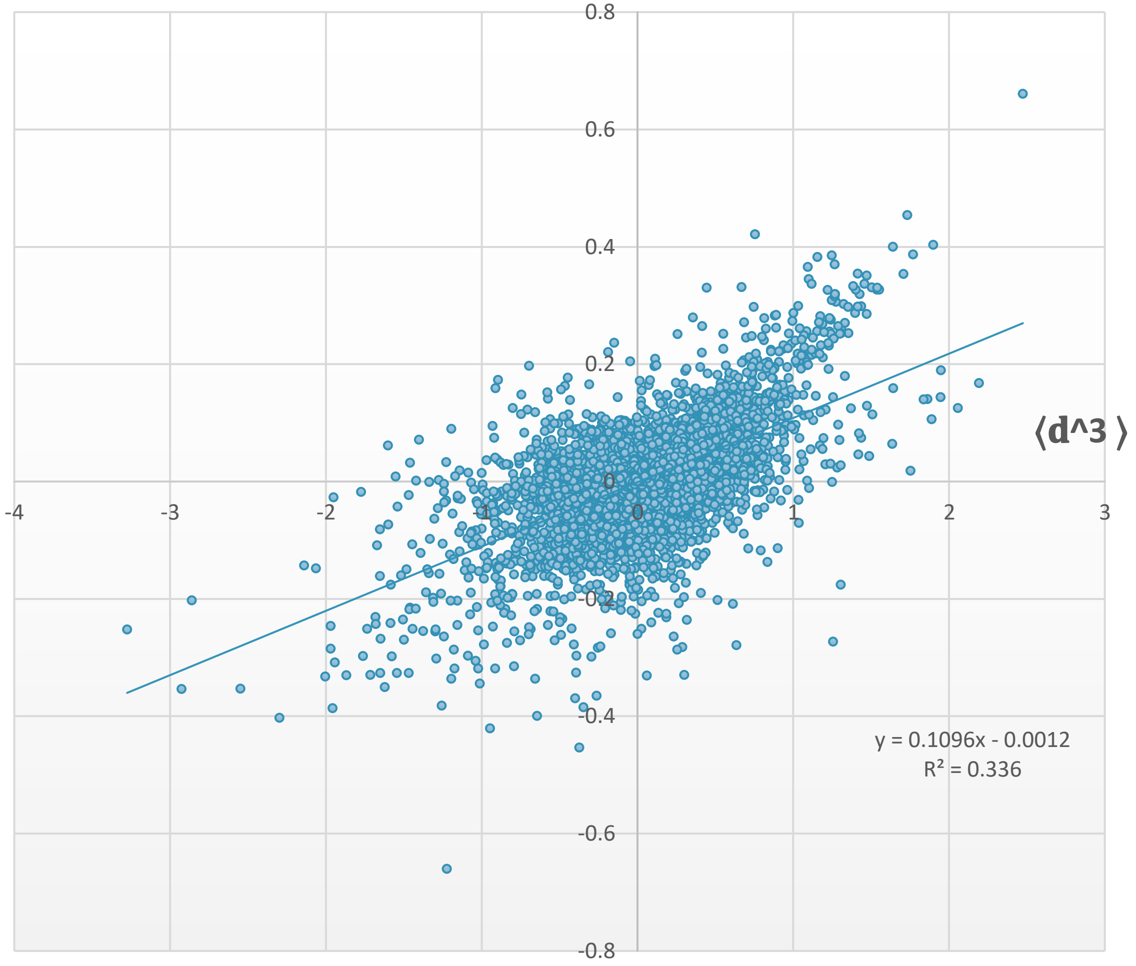
Spatial (ICP) price indices: "skewness"

Now, let's look at the spatial indices.

We built the same graph for binary indices. The first thing we learn from the graph is that the relationship becomes much fuzzier (r^2=0.34). The fuzziness is attributed not only to the uncertainty of prices (price shocks of different directions) but also to the fact that there is also no such single dominant item as tech in the NIPA temporal series. Besides, tech price declines apply to all countries simultaneously and thus are not relevant for the spatial case. We see that $\langle \mathbf{d}^3 \rangle$ in the ICP case is extremely high – up to 3 and more, compared to 0.09 in the temporal NIPA case w/o tech items (Figure 9). Thus, we can conclude that the ICP case should be almost completely described with the shock equations.

We know that the effects of price shocks are symmetric (shocks of opposite directions of the same magnitude and weight get canceled out), hence we can conclude that these price shocks in ICP are non-symmetric, which points to the possibility of serious biases due to data errors.

Fig. 12. Fisher-Törnqvist ratio vs. ⟨d^3 ⟩, 2021 ICP, binary



Thus, the resulting effect of multiple price shocks can be described by the sum of models described by expression (2) (when **s** and **r** are small).

$$log\ F(r,s) - \ log\ T(r,s) \approx \sum \frac{s_k}{6} {r_k}^3 \qquad (11)$$

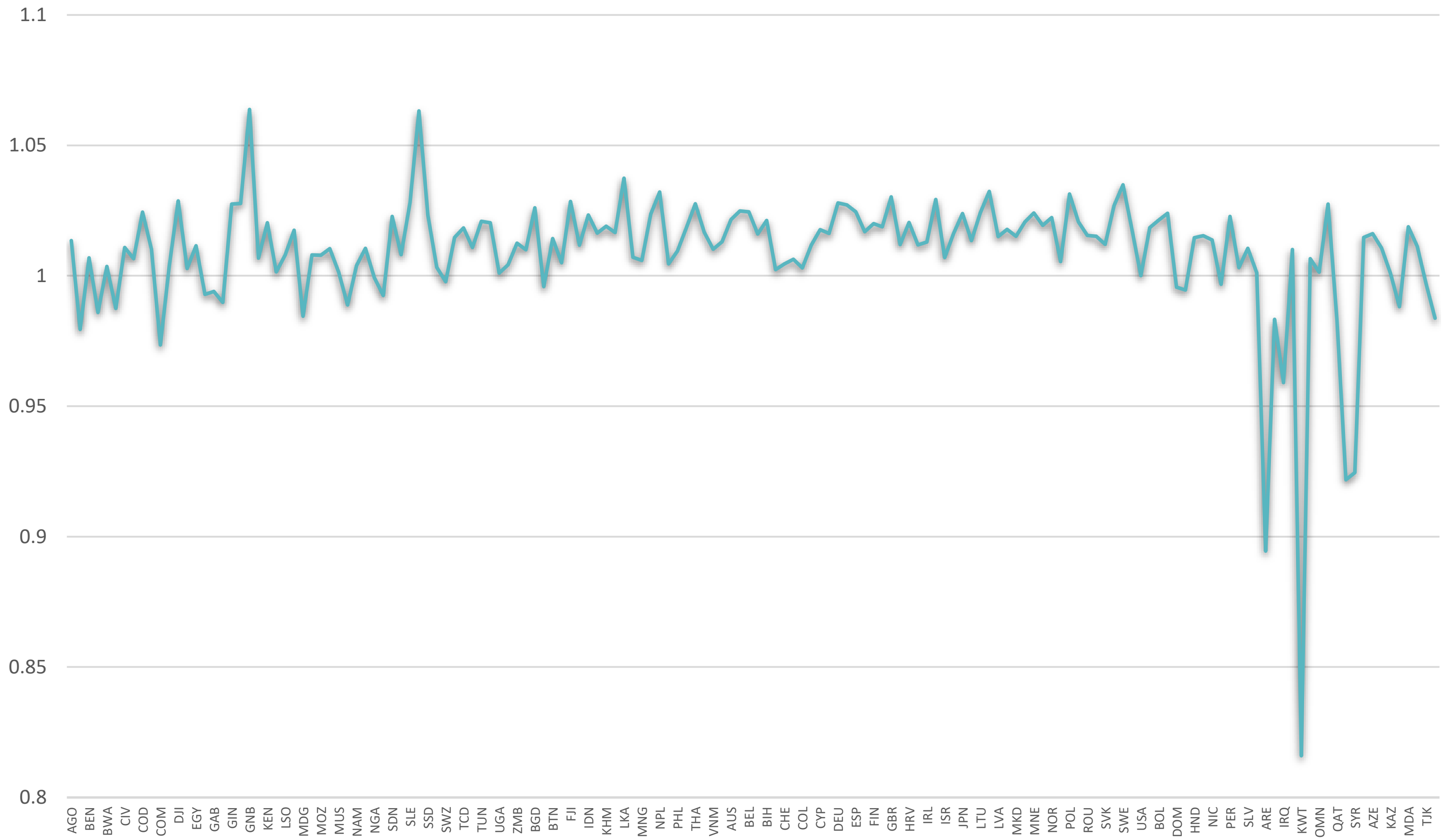
It is clear, that the uncertainty in expenditure weights has substantial influence as well.

Fig. 5. Fisher-Törnqvist differences (in logs, absolute value), matrix, by region, by country, 154 ICP countries, 2021 PCE w/o Net Expenditure Abroad

legend for colors:
black > 0.40; red > 0.20, orange > 0.10, grey > 0.05, no color < 0.05

Spatial price indices

Fig. 6. GEKS/CCD PPP ratios, ICP 2021 global unrestricted results, US = 1, by country



1. Explaining the ICP discrepancies with the model of price shocks (errors)

Now we will apply the same model to the ICP (spatial comparison). First, we notice that we cannot know the "true" index, and all we can do is to look at the GEKS/CCD and Fisher/Törnqvist differences. But what we do know is that the GEKS and Fisher do amplify distortions due to shocks. The task at hand is immensely complicated by an extremely high uncertainty of the price relatives – the PPPs (as well as an equally distortive uncertainty in expenditure shares, but this is a story for another paper). Indirectly, the PPP volatility can be inferred from the temporal volatility of the BH PPPs which we can compare against our expectations.

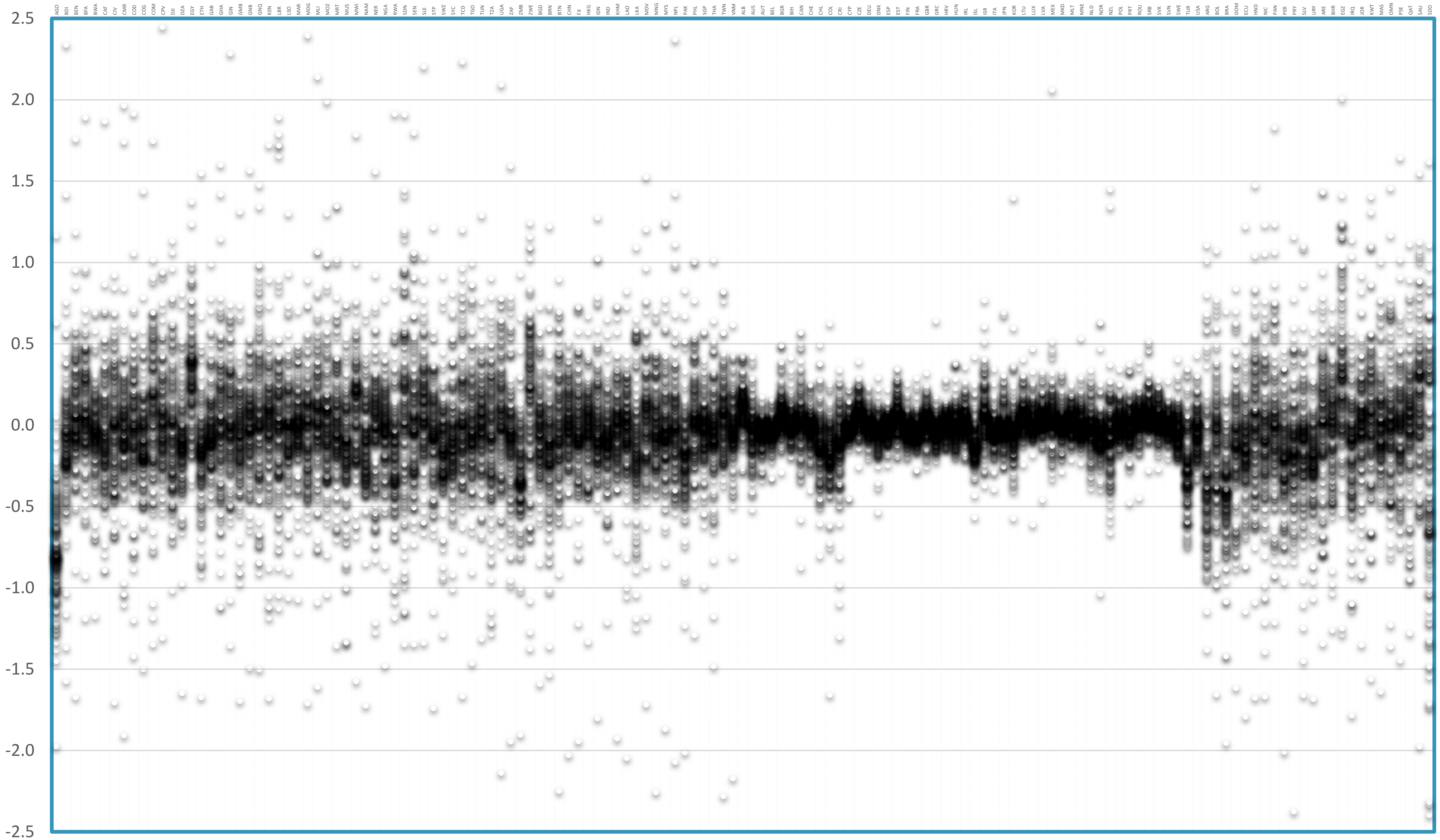
The US NIPA PCE data we work with contain 72 components which is not too far off from the ICP BH count for the same aggregate – 109. The 2017-2021 PCE component volatility in the US was less than 10% for those 72 components. In the international setting we expect many components (BHs) to behave in a correlated manner across countries, esp. those traded internationally such as cars, electronics, fuel, etc. Thus, if we assume that within, say, the OECD the markets are functioning reasonably well, we can expect temporal volatility of US prices relative to the OCED average (or, PPP(US, 2017)/PPP(OECD, 2017) vs. PPP(US, 2021)/PPP(OECD, 2021)) to be even lower than 10% (probably significantly lower). In reality, it is 13.9% (in logs) which is still not too bad, and this probably means that the average BH PPP error (uncertainty) for the US in the OECD comparison is about 10% or so.

1. Explaining the ICP discrepancies with the model of price shocks (errors)

Unfortunately, it is worse for many other countries: the chart below shows the 2021/2017 PPP changes by country (shown in PLI form in order to exclude local inflation). The chart below is presented in logs, thus the scale spans from -2.5 to +2.5, or 148 times (148 = Exp(5)). This is quite significant as a price jolt of that magnitude can seriously distort the aggregate PPPs.

It would be fair to say that the data points beyond the 20% band around the midpoint on this graph probably indicate price measurement errors. Which are significant.

Fig. 8. PLI changes by BH, by country, 2021 to 2017, OECD=0.0, ICP 2021, in logs



Conclusions

This paper finds that the Törnqvist-based CCD index is strongly preferrable to the Fisher-based GEKS index under the condition of strong price and expenditure shocks [uncertainties]. So how does it correspond to the results of Dikhanov (2024) that finds the Walsh index “ideal”?

In short, that paper finds that the range of the best superlative indices lies between the Fisher and *implicit* Törnqvist, with the Walsh being lodged comfortably in the middle [see Figure 6 of Dikhanov (2024) reproduced below, the Walsh index is in green].

Some notation (from paper #2)

The arithmetic-harmonic weighted power mean describes a generalized price index:

$$GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \left(\sum \mathbf{s}_{0i} x_i^{\alpha} \Big/ \sum \mathbf{s}_{1i} x_i^{-\alpha}\right)^{\frac{1}{2\alpha}} \qquad (1)$$

Expression (1) describes some well-known superlative price indices, such as the Fisher ($\alpha = 1$), Törnqvist ($\alpha = 0$), and Diewert ($\alpha = ½$) . The Lloyd-Moulton index can be presented using expression (1) also, and so some elementary (i.e., unweighted) indices. The Walsh index is not part of this index class.

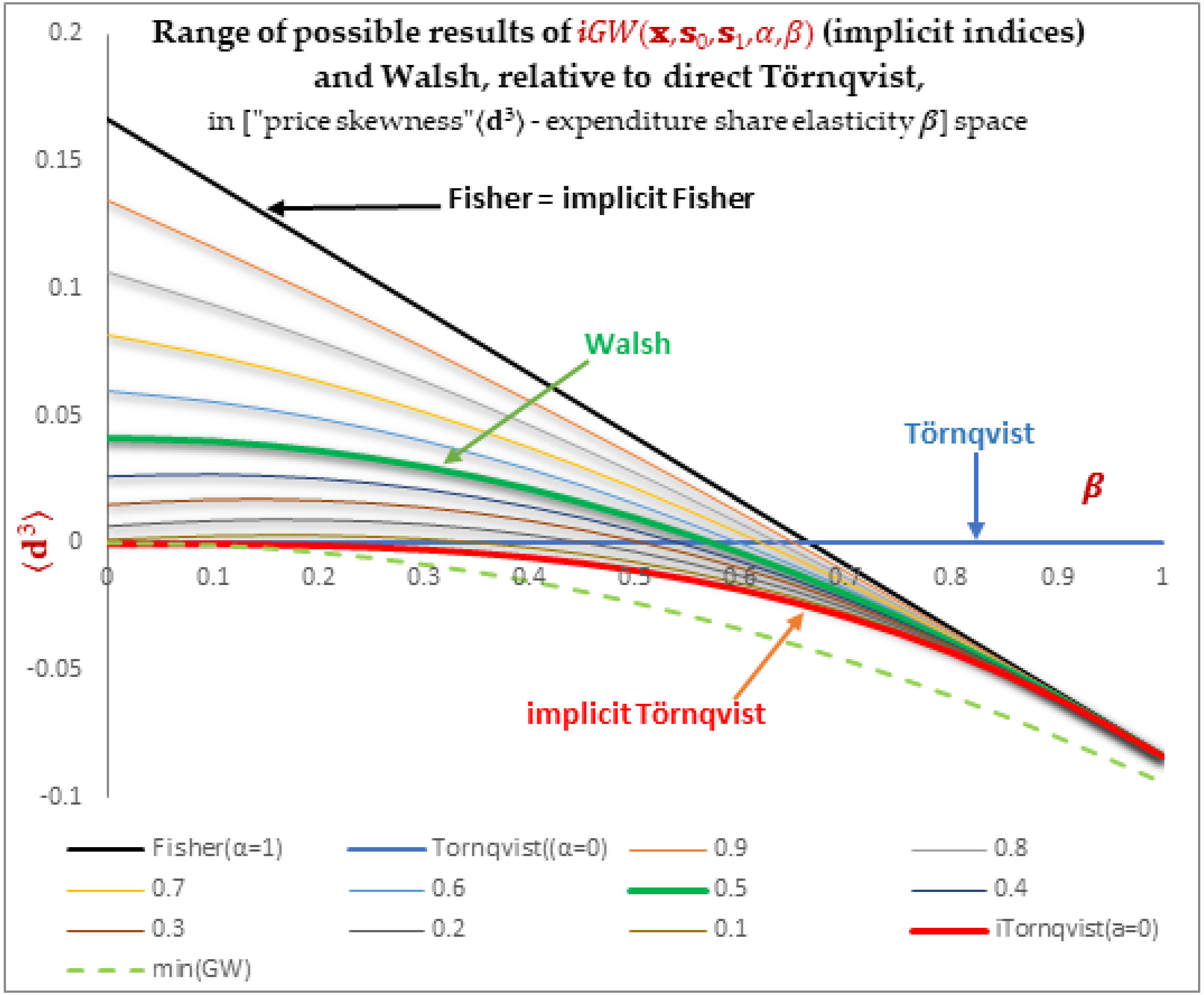
The following functional form describes the implicit price index corresponding to expression (1) for the direct price index $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha)$:

$$iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \sum s_{0i} x_i y_i \Big/ \left(\sum s_{0i} y_i^{\alpha} \Big/ \sum s_{1i} y_i^{-\alpha}\right)^{\frac{1}{2\alpha}} \qquad (27)$$

Generalized index $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ discussed in the previous sections does not include the Walsh index. However, it is possible to construct an index that would go through the Fisher ($\alpha = 1$) and Törnqvist ($\alpha = 0$) indices in the limiting cases, and, simultaneously, pass through the Walsh index ($\alpha = ½$). Let's call it *GW2* and define it as follows:

$$GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \left(\sum \mathbf{s}_{0i}{}^{\alpha} \mathbf{s}_{1i}{}^{1-\alpha} x_i^{\alpha} \Big/ \sum \mathbf{s}_{0i}{}^{1-\alpha} \mathbf{s}_{1i}{}^{\alpha} x_i{}^{-\alpha}\right)^{\frac{1}{2\alpha}} \qquad (42)$$

And setting α = 0, ½ and 1, we arrive at the Törnqvist (expression (3)), Walsh (expression (7)) and Fisher (expression (5)) indices, respectively.



Range of possible results of $iGW(\mathbf{x},\mathbf{s}_0,\mathbf{s}_1,\alpha,\beta)$ (implicit indices)
and Walsh, relative to direct Törnqvist,
in ["price skewness"$\langle \mathbf{d}^3 \rangle$ - expenditure share elasticity β] space
Fisher = implicit Fisher
Walsh
Törnqvist
β
implicit Törnqvist
$\langle \mathbf{d}^3 \rangle$
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Fisher(α=1)
Tornqvist((α=0)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
iTornqvist(a=0)
min(GW)

First, the Walsh was found in that paper to be the best for the whole interval of $\beta \subset [0,1]$, and until β=2/3 the regular Törnqvist was within that range of the best superlative indices, thus for most reasonable cases of elasticity β the Törnqvist is still quite good. Another consideration is the following: that paper assumes measurement errors to be small. However, when shocks [measurement errors] start contributing significantly (and, especially, overwhelmingly) to $\langle \mathbf{d}^3 \rangle$ (i.e., to the "skewness" of price variations), the situation changes dramatically, and the index stability considerations become extremely important.

Perhaps, when the ICP price uncertainties diminish in the future ICP rounds we could consider the Walsh-based EKS index [we can call it W-EKS] for PPP calculations. Until then, the CCD index will be the best index we can get for the ICP under the circumstances.

A bonus: the GEKS computation can collapse if some weights become negative, not the CCD.

Thank you!