

Range of possible results for certain classes of superlative price indices

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This paper aims to find the range of possible results for certain classes of well-known price indices, both unweighted (elementary) and weighted, for an arbitrary smooth weight function. The classes of indices include the Fisher, Tornqvist, Walsh, Diewert, Lloyd-Moulton and other indices. Both direct and implicit price indices are discussed. In general, approximations of the third order (and, in one case, those of the fifth order) are employed. The range of possible results is expressed in relationship to the geometric Laspeyres index. Stability of the indices is discussed and some recommendations for the “best” index are given.

The arithmetic-harmonic weighted power mean describes a generalized price index:

$$GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \left(\sum \mathbf{s}_{0i} x_i^\alpha / \sum \mathbf{s}_{1i} x_i^{-\alpha} \right)^{\frac{1}{2\alpha}} \quad (1)$$

Here all sums and products refer to the sums and products with subscripts running from 1 to N: $\sum x_i = \sum_{i=1}^N x_i$ and $\prod x_i = \prod_{i=1}^N x_i$, where N is the number of products.

Consider the following transformation: $x_i = \frac{p_i^1}{p_i^0} / \prod \left(\frac{p_i^1}{p_i^0} \right)^{s_{0i}}$. Here $\prod \left(\frac{p_i^1}{p_i^0} \right)^{s_{0i}}$ is the Geometric Laspeyres index. Henceforth, all indices in this paper are presented in terms of the Geometric Laspeyres index, which is equal to one under this transformation: $\prod x_i^{s_{0i}} = 1$.

Expression (1) describes some well-known superlative price indices, such as the Fisher ($\alpha = 1$), Törnqvist ($\alpha = 0$), and Diewert ($\alpha = 1/2$) [the Diewert index is described in Diewert (1976)]. The Lloyd-Moulton index can be presented using expression (1) also, and so some elementary (i.e., unweighted) indices. However, the Walsh index is not part of this index class.

Generalized Weighted Index

Let's assume that item shares change with prices with elasticity β : $s_{1i} = s_{0i}e^{\beta d_i} / \sum s_{0i}e^{\beta d_i}$, where $d_i = \log x_i$. Then, we can rewrite expression (1) as:

$$\begin{aligned}
\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) &= \frac{1}{2\alpha} \log \left(\sum s_{0i} x_i^\alpha / \sum s_{1i} x_i^{-\alpha} \right) \\
&= \frac{1}{2\alpha} \left(\log \sum s_{0i} e^{\alpha d_i} - \log \sum s_{0i} e^{\beta d_i} e^{-\alpha d_i} + \log \sum s_{0i} e^{\beta d_i} \right) \approx \\
&\approx \frac{1}{2\alpha} \left(\log \sum s_{0i} \left(1 + \alpha d_i + \frac{\alpha^2 d_i^2}{2} + \frac{\alpha^3 d_i^3}{6} \right) \right. \\
&\quad \left. - \log \sum s_{0i} \left(1 + (\beta - \alpha) d_i + \frac{(\beta - \alpha)^2 d_i^2}{2} + \frac{(\beta - \alpha)^3 d_i^3}{6} \right) \right. \\
&\quad \left. + \log \sum s_{0i} \left(1 + \beta d_i + \frac{\beta^2 d_i^2}{2} + \frac{\beta^3 d_i^3}{6} \right) \right)
\end{aligned}$$

And noting that $\sum s_{0i} d_i = 0$ by construction, we get:

$$\begin{aligned}
\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) &\approx \frac{1}{2\alpha} \left(\log \left(1 + \sum s_{0i} \left(\frac{\alpha^2 d_i^2}{2} + \frac{\alpha^3 d_i^3}{6} \right) \right) - \log \left(1 + \sum s_{0i} \left(\frac{\beta^2 d_i^2}{2} + \frac{\alpha^2 d_i^2}{2} - \alpha \beta d_i^2 + \frac{\beta^3 d_i^3}{6} - \frac{\alpha^3 d_i^3}{6} + \frac{\alpha^2 \beta d_i^3}{2} - \frac{\alpha \beta^2 d_i^3}{2} \right) \right) \right. \\
&\quad \left. + \log \left(1 + \sum s_{0i} \left(\frac{\beta^2 d_i^2}{2} + \frac{\beta^3 d_i^3}{6} \right) \right) \right)
\end{aligned}$$

Applying Taylor expansion for logarithm, $\log(1+x) \approx x - x^2/2 + x^3/3$ [in our case only one term of the expansion is used, as the second term already produces d_i^4 and above], and discarding terms higher than the third order, we get the following (utilizing the moments' notation $\langle \mathbf{d}^n \rangle = \sum s_{0i} d_i^n$):

$$\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\alpha^2}{6} - \frac{\alpha\beta}{4} + \frac{\beta^2}{4} \right) \langle \mathbf{d}^3 \rangle \quad (2)$$

Törnqvist, Fisher and Diewert Indices

From expression (2) we directly obtain the approximation for the Törnqvist index by setting $\alpha = 0$:

$$\log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \frac{\beta^2}{4} \langle \mathbf{d}^3 \rangle \quad (3)$$

By setting $\alpha = 1/2$ we get the approximation for the Diewert index:

$$\log D(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\beta^2}{4} + \frac{1}{24} - \frac{\beta}{8} \right) \langle \mathbf{d}^3 \rangle \quad (4)$$

Accordingly, by setting $\alpha = 1$ we obtain the approximation for the Fisher index:

$$\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\beta^2}{4} + \frac{1}{6} - \frac{\beta}{4} \right) \langle \mathbf{d}^3 \rangle \quad (5)$$

Lloyd-Moulton Index

The Lloyd-Moulton index is written as:

$$LM(\mathbf{x}, \mathbf{s}_0, \sigma) = \left(\sum \mathbf{s}_{0i} x_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Or, utilizing elasticity $\beta = 1 - \sigma$:

$$LM(\mathbf{x}, \mathbf{s}_0, \beta) = \left(\sum \mathbf{s}_{0i} x_i^\beta \right)^{\frac{1}{\beta}}$$

There is also the “backward” LM index described as:

$$LMb(\mathbf{x}, \mathbf{s}_1, \beta) = \left(\sum \mathbf{s}_{1i} x_i^{-\beta} \right)^{-\frac{1}{\beta}}$$

It is easy to see that under the condition that $\mathbf{s}_{1i} = \mathbf{s}_{0i} e^{\beta d_i} / \sum \mathbf{s}_{0i} e^{\beta d_i}$, $LM(\mathbf{x}, \mathbf{s}_0, \beta) = LMb(\mathbf{x}, \mathbf{s}_1, \beta)$. Thus, combining the two will produce $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta, \beta)$. Therefore, from expression (2), by setting $\alpha = \beta$ we get the following approximation for the Lloyd-Moulton index:

$$\log LM(\mathbf{x}, \mathbf{s}_0, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \frac{\beta^2}{6} \langle \mathbf{d}^3 \rangle \quad (6)$$

U.S.BLS uses the Lloyd-Moulton index with $\beta \approx 0.4$ in their preliminary estimates of the CPI with the target of predicting the Törnqvist index. From expressions (3) and (6) we see that the misprediction with the Lloyd-Moulton index in this case is $-\frac{\beta^2}{12} \langle \mathbf{d}^3 \rangle$ (from expressions (3) and (6)).

Walsh index

The Walsh index cannot be approximated using expression (2). Thus, we need to do it separately (note that $\sum \mathbf{s}_{0i} e^{\beta d_i}$ in the derivation cancels out):

$$\begin{aligned}
\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) &= \log \left(\sum (s_{0i}s_{1i})^{1/2} x_i^{1/2} / \sum (s_{0i}s_{1i})^{1/2} x_i^{-1/2} \right) \\
&= \log \sum (s_{0i}s_{1i})^{1/2} e^{d_i/2} - \log \sum (s_{0i}s_{1i})^{1/2} e^{-d_i/2} \\
&= \log \sum s_{0i} e^{\beta d_i/2} e^{d_i/2} - \log \sum s_{0i} e^{\beta d_i/2} e^{-d_i/2}
\end{aligned}$$

Or applying Taylor expansion for the exponent and noting that $\sum s_{0i} d_i = 0$ by construction, we get:

$$\begin{aligned}
\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) &\approx \\
&\approx \log \sum s_{0i} \left(1 + \frac{(\beta+1)d_i}{2} + \frac{(\beta+1)^2 d_i^2}{8} + \frac{(\beta+1)^3 d_i^3}{48} \right) \\
&- \log \sum s_{0i} \left(1 + \frac{(\beta-1)d_i}{2} + \frac{(\beta-1)^2 d_i^2}{8} + \frac{(\beta-1)^3 d_i^3}{48} \right) \\
&\approx \log \left(1 + \sum s_{0i} \left(\frac{(\beta+1)^2 d_i^2}{8} + \frac{(\beta+1)^3 d_i^3}{48} \right) \right) - \log \left(1 + \sum s_{0i} \left(\frac{(\beta-1)^2 d_i^2}{8} + \frac{(\beta-1)^3 d_i^3}{48} \right) \right)
\end{aligned}$$

And applying Taylor expansion for logs we get:

$$\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{(\beta+1)^2}{8} \langle \mathbf{d}^2 \rangle + \frac{(\beta+1)^3}{48} \langle \mathbf{d}^3 \rangle - \frac{(\beta-1)^2}{8} \langle \mathbf{d}^2 \rangle - \frac{(\beta-1)^3}{48} \langle \mathbf{d}^3 \rangle$$

Finally, we obtain:

$$\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{1}{24} + \frac{\beta^2}{8} \right) \langle \mathbf{d}^3 \rangle \quad (7)$$

Elementary indices (Jevons, Dikhanov, CSWD)

The elementary (unweighted) indices can be approximated using expression (2) as well. Assuming that $s_{1i} = s_{0i}$, or $\beta = 0$, we get the following expressions for the elementary indices:

$$\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, 0) \approx \frac{\alpha^2}{6} \langle \mathbf{d}^3 \rangle \quad (8)$$

In particular,

Jevons ($\alpha = 0$):

$$\log Je(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, 0, 0) = 0 \quad (9)$$

Dikhanov ($\alpha = 0.5$):

$$\log De(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, 0.5, 0) \approx \frac{1}{24} \langle \mathbf{d}^3 \rangle \quad (10)$$

CSWD ($\alpha = 1$):

$$\log Ce(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, 1, 0) \approx \frac{1}{6} \langle \mathbf{d}^3 \rangle \quad (11)$$

Note that expressions (8) – (11) are correct in the general case $s_{1i} = s_{0i}$ (or when item shares are constant between the two periods), so they will be correct in the special case $s_{1i} = s_{0i} = 1/N$ as well. From expressions (9) – (11) it follows that the Dikhanov index is located between the Jevons and CSWD indices in a one-to-three relationship.

Range of possible results of Generalized Power Index $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$

Let us combine expressions (2) – (11) to build up the range of possible results for the function $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ and the Walsh index presented here. We will express all indices relative to the Törnqvist index. Thus, the difference between $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ and the Tornqvist becomes:

$$\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{\alpha^2}{6} - \frac{\alpha\beta}{4} \right) \langle \mathbf{d}^3 \rangle \quad (12)$$

For the Walsh, Diewert, Fisher, and Lloyd-Moulton indices, the difference becomes, respectively:

$$\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{1}{24} - \frac{\beta^2}{8} \right) \langle \mathbf{d}^3 \rangle \quad (13)$$

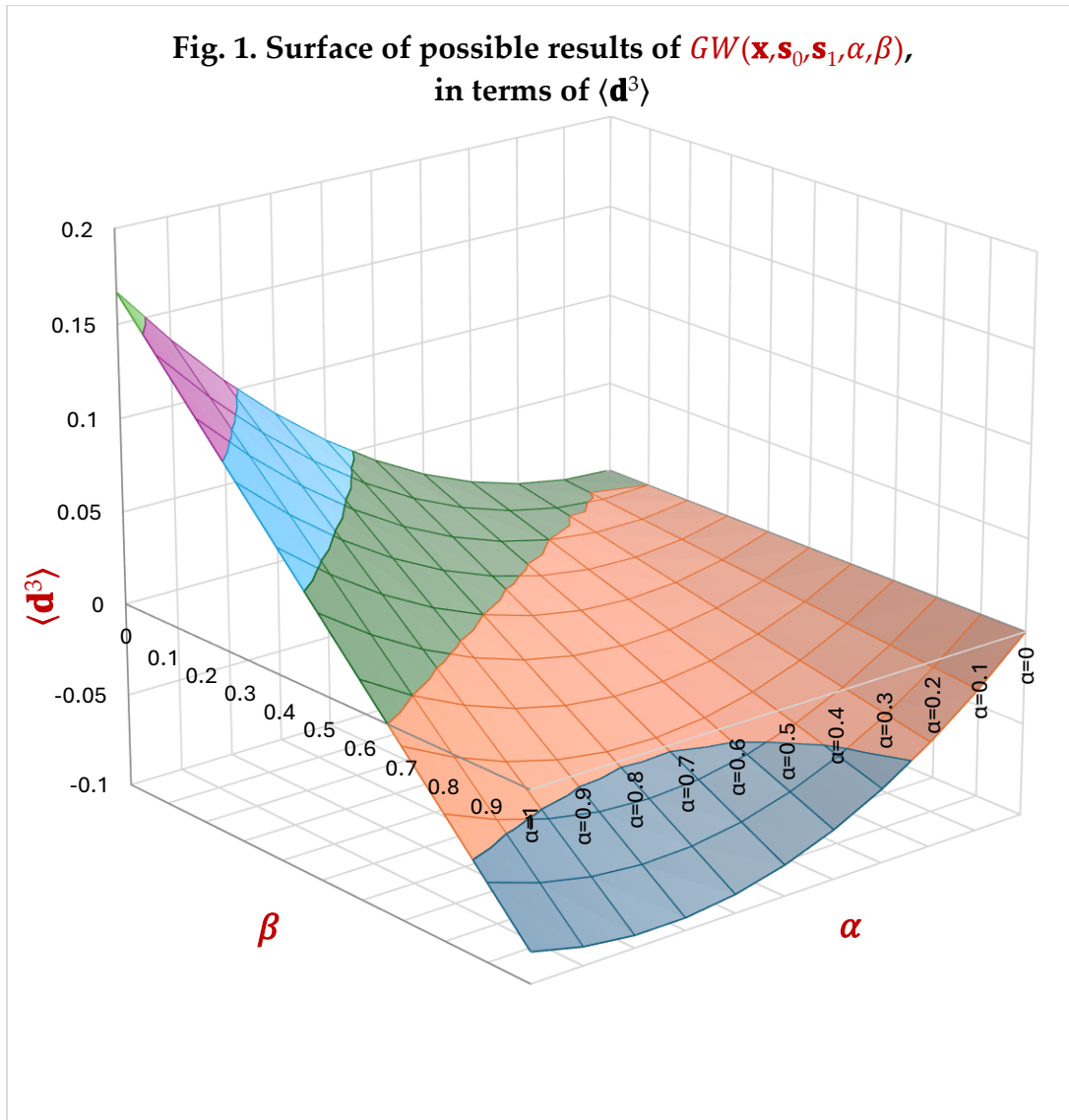
$$\log D(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{1}{24} - \frac{\beta}{8} \right) \langle \mathbf{d}^3 \rangle \quad (14)$$

$$\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{1}{6} - \frac{\beta}{4} \right) \langle \mathbf{d}^3 \rangle \quad (15)$$

$$\log LM(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(-\frac{\beta^2}{12} \right) \langle \mathbf{d}^3 \rangle \quad (16)$$

The expressions for the elementary indices do not change, as the Tornqvist becomes the Jevons index when $\alpha = 0$ and $\beta = 0$. From expressions (13) – (16) it follows that the indices described by expressions (8) – (11) are degenerate cases (i.e., the unweighted varieties) of the weighted indices described by expressions (12) – (15). Note that the Dikhanov index ($\alpha = 0.5$) is the unweighted variant of both the Diewert and Walsh indices. The CSWD index ($\alpha = 1$) is the unweighted variant of the Fisher index, and the Jevons index ($\alpha = 0$) is the unweighted variant of the Törnqvist index.

Let's define the boundaries of the range under consideration. We will vary α from 0 to 1, and β from 0 to 1. Those ranges will cover indices from the Tornqvist ($\alpha = 0$) to the Fisher ($\alpha = 1$), on the α dimension, and the cases from elasticity ($\beta=0$) to elasticity ($\beta=1$)¹; with everything else in between. The surface of possible results in the 3D space is presented in Fig. 1 below.



¹ From the U.S.BLS experience, the most plausible values of β lie between 0.3 and 0.5 at the aggregate level.

Now, consider the projection of this surface onto the $[\langle \mathbf{d}^3 \rangle - \beta]$ plane. Note that the minimum of the projected surface is achieved at $\alpha = 3\beta/4$ (we find it by differentiating expression (12)) where $\min(\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)) \approx -3\beta^2/32$.

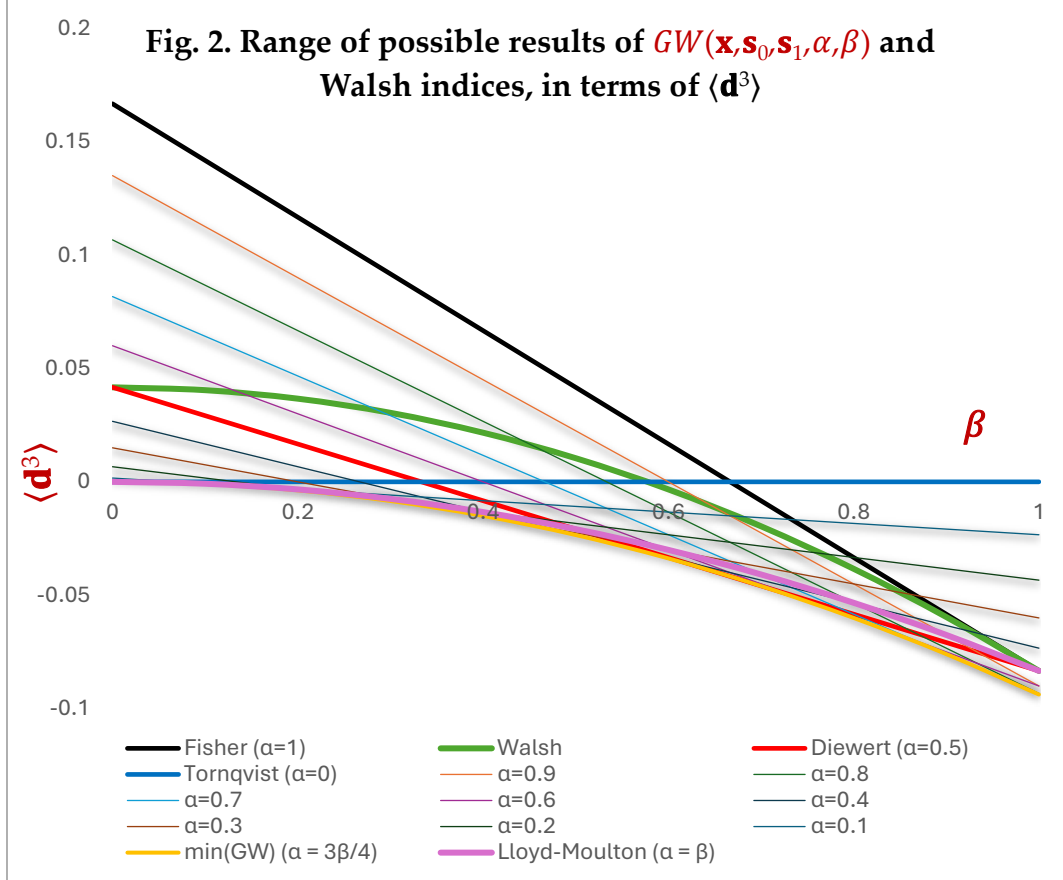
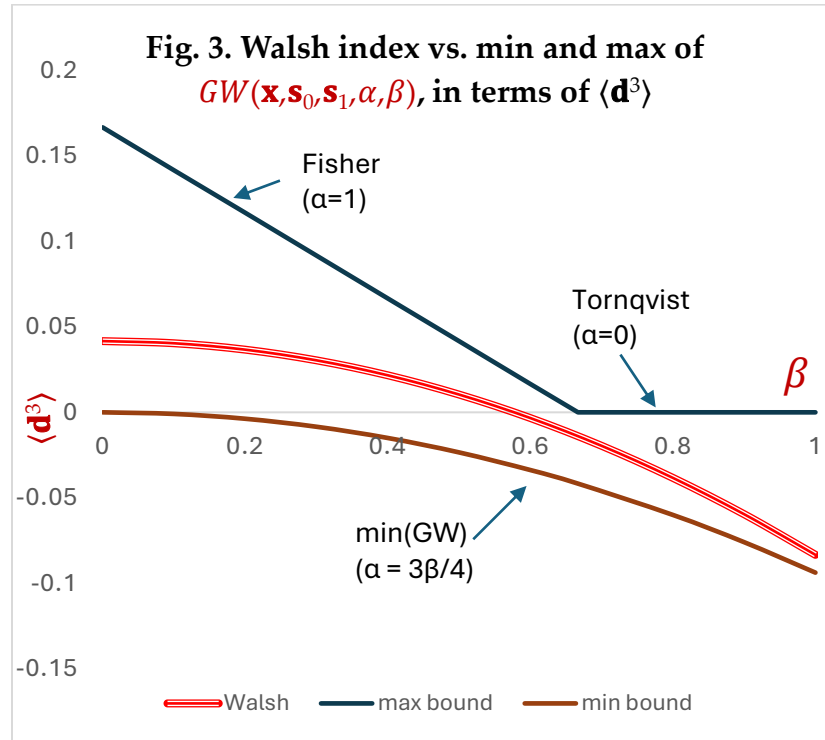


Figure 2 shows all indices investigated in this paper as well as the general behavior of $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ with α ranging from 0 to 1. We see that as power α increases, elasticity β becomes more discriminating for the indices with the Fisher index ($\alpha = 1$) exhibiting the highest degree of variation. In this graph we also show the Walsh index described by expression (13).

The Walsh index approaches the Diewert index when $\beta \rightarrow 0$ and $\beta \rightarrow 1$, and, as mentioned earlier, its unweighted counterpart is the same index as for the Diewert. The Diewert index approaches the Törnqvist when $\beta \rightarrow 1/3$, for the Walsh this value is $1/\sqrt{3}$, and for the Fisher, it is $2/3$. The Lloyd-Moulton index approaches the Tornqvist index as $\beta \rightarrow 0$. The four indices [Walsh, Diewert, Fisher, and Lloyd-Moulton] converge to $-\frac{1}{12}\langle \mathbf{d}^3 \rangle$ when $\beta > 1$, relative to the Törnqvist index. It also follows that the indices can change their order

[relative position], as their order depends on the sign of $\langle \mathbf{d}^3 \rangle$ and the value of β . For example, the Walsh index is located between the Fisher and Tornqvist indices when $\beta < 1/\sqrt{3}$, at the same time the Walsh is located between the Diewert and Fisher indices when $\beta \in [0, 1]$. As $\beta \rightarrow 0$, the Tornqvist index approaches the Geometric Laspeyres, the Fisher approaches $\frac{1}{6}\langle \mathbf{d}^3 \rangle$, and both the Walsh and Diewert approach $\frac{1}{24}\langle \mathbf{d}^3 \rangle$. Note that this corresponds to expression (8) for unweighted indices with α being equal to 1 and 0.5, respectively. Thus, when $\beta \rightarrow 0$ the weighted indices mirror the behavior of their unweighted counterparts.

Figure 3 below depicts the bounds of $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ and the Walsh index. The Fisher index determines the upper bound (until $\beta = 2/3$), then the Törnqvist index determines the upper bound. The lower bound is achieved with $\alpha = 3\beta/4$ (note the behavior of the Lloyd-Moulton index ($\alpha = \beta$) located somewhat above the lower bound, see Figure 2). The Walsh index is located in between the bounds of $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ for all β .

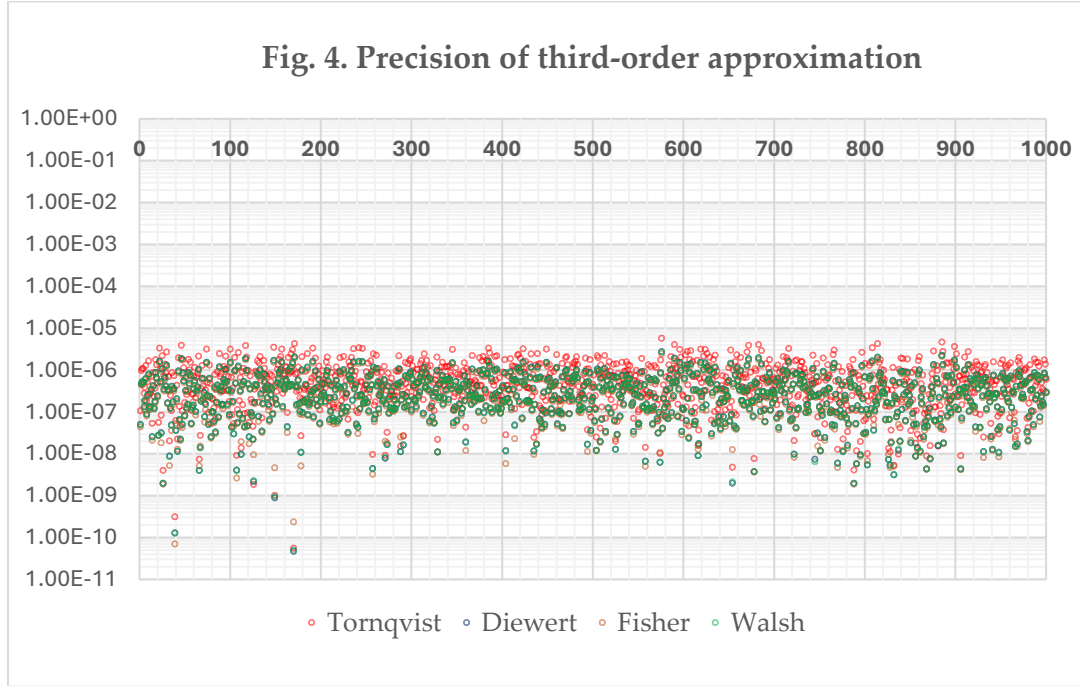


Precision of the third-order approximation

Precision evaluation of expressions (2) – (11) was carried out with the Monte Carlo process. To this end, we set up a series of 1,000 simulations with β ranging from 0 to 1, and $r = 0.25$. Prices were simulated with process $\log \mathbf{x} = r^* \mathbf{U}[0,1]$ (uniform distribution).

Weights were simulated with process $\mathbf{s} = \mathbf{U}[0,1]$, then normalized for their sum to be equal to one.

Results of the simulation: The third-order approximations ensure high precision results, with the worst approximation exceeding a five-digit accuracy, and with the average error being as low as in the seventh digit. That is more than adequate for any practical purposes.



Some practical considerations for the Consumer Price Index

The sign of $\langle \mathbf{d}^3 \rangle$ is mostly negative, as we have a few technology products whose prices decline (relatively or absolutely) against the background of mostly rising prices for the rest². Therefore, the combination of Fisher and Tornqvist in this case represents the absolute minimum of $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ (with a switching point at $\beta = 2/3$).

$GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ will be positioned above the combination of these two indices at all points. Thus, the Walsh index may look attractive in this setting as it is placed right in the middle of all possible results of $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$, never approaching its bounds.

² This consideration applies only to temporal indices. For spatial indices the technical change responsible for relative price declines of technology items would be applicable to all countries/ territories simultaneously.

Another practical implication of expression (12) is a straightforward derivation of elasticity β . From $\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx (\frac{1}{6} - \frac{\beta}{4})\langle \mathbf{d}^3 \rangle$ we get:
 $\beta \approx 4((\log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta))/\langle \mathbf{d}^3 \rangle + \frac{1}{6})$.

Extension of GW to arbitrary weight function

From Dikhanov (2024, <https://stats.unece.org/ottawagroup/download/Dikhanov-A-New-Elementary-Index.pdf>) it follows that the expressions (8) - (16) [the differences between various indices and the Törnqvist] are valid for the weight function $s_{1i} = s_{0i} (1 + \beta d_i)$ as well. The only difference exists in expressions (2) - (7) where an extra term $(\frac{\beta^2}{4}\langle \mathbf{d}^3 \rangle)$ is present but that doesn't affect the differences *between* the indices and the Törnqvist.

Now, consider a general weight functional form $s_{1i} = s_{0i} f(d_i)/\sum s_{0i} f(d_i)$ where $f(d_i)$ is smooth three-times continuously differentiable function, and $f(0) = 1$ at $d_i = 0$.

Generalized Weighted Index (extended)

Now we can rewrite expression (1) with $s_{1i} = s_{0i} f(d_i)/\sum s_{0i} f(d_i)$ in mind as:

$$\begin{aligned} \log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.)) &= \frac{1}{2\alpha} \log \left(\frac{\sum s_{0i} x_i^\alpha}{\sum s_{1i} x_i^{-\alpha}} \right) = \\ &= \frac{1}{2\alpha} (\log \sum s_{0i} e^{\alpha d_i} - \log \sum s_{0i} f(d_i) e^{-\alpha d_i} + \log \sum s_{0i} f(d_i)) \end{aligned}$$

Using Taylor (actually, Maclaurin) expansion $f(d_i) \approx f(0) + f'(0)d_i + 1/2f''(0)d_i^2 + 1/6f'''(0)d_i^3$, we get:

$$\begin{aligned} \log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.)) &\approx \\ &\approx \frac{1}{2\alpha} (\log \sum s_{0i} \left(1 + \alpha d_i + \frac{\alpha^2 d_i^2}{2} + \frac{\alpha^3 d_i^3}{6} \right) \\ &- \log \sum s_{0i} \left(f(0) + f'(0)d_i + \frac{f''(0)d_i^2}{2} + \frac{f'''(0)d_i^3}{6} \right) \left(1 - \alpha d_i + \frac{\alpha^2 d_i^2}{2} \right. \\ &\left. - \frac{\alpha^3 d_i^3}{6} \right) + \log \sum s_{0i} (f(0) + f'(0)d_i + \frac{f''(0)d_i^2}{2} + \frac{f'''(0)d_i^3}{6})) \end{aligned}$$

And noting that $\sum s_{0i} d_i = 0$ by construction, we obtain:

$$\begin{aligned} \log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.)) \\ \approx \frac{1}{2\alpha} (\log (1 + \sum s_{0i} \left(\frac{\alpha^2 d_i^2}{2} + \frac{\alpha^3 d_i^3}{6} \right)) \\ - \log \left(\sum s_{0i} \left(1 + f'(0)d_i + \frac{f''(0)d_i^2}{2} + \frac{f'''(0)d_i^3}{6} \right) \left(1 - \alpha d_i + \frac{\alpha^2 d_i^2}{2} \right. \right. \\ \left. \left. - \frac{\alpha^3 d_i^3}{6} \right) \right) + \log (1 + \sum s_{0i} \left(\frac{f''(0)d_i^2}{2} + \frac{f'''(0)d_i^3}{6} \right))) \end{aligned}$$

Noting that $\sum s_{0i} f'(0)d_i = 0$,

$$\begin{aligned} \log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.)) \\ \approx \frac{1}{2\alpha} \left(\sum s_{0i} \left(\frac{\alpha^2 d_i^2}{2} + \frac{\alpha^3 d_i^3}{6} \right) \right. \\ \left. - \sum s_{0i} \left(\frac{f''(0)d_i^2}{2} + \frac{f'''(0)d_i^3}{6} - \alpha f'(0)d_i^2 - \frac{\alpha f''(0)d_i^3}{2} + \frac{\alpha^2 d_i^2}{2} \right. \right. \\ \left. \left. + \frac{\alpha^2 f'(0)d_i^3}{2} - \frac{\alpha^3 d_i^3}{6} \right) + \sum s_{0i} \left(\frac{f''(0)d_i^2}{2} + \frac{f'''(0)d_i^3}{6} \right) \right) \end{aligned}$$

Or,

$$\begin{aligned} \log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.)) \\ \approx \frac{1}{2\alpha} \left(\sum s_{0i} \left(\frac{\alpha^2 d_i^2}{2} + \frac{\alpha^3 d_i^3}{6} \right) \right. \\ \left. - \sum s_{0i} \left(-\alpha f'(0)d_i^2 - \frac{\alpha f''(0)d_i^3}{2} + \frac{\alpha^2 d_i^2}{2} + \frac{\alpha^2 f'(0)d_i^3}{2} - \frac{\alpha^3 d_i^3}{6} \right) \right) \end{aligned}$$

Finally, collecting the terms, we obtain:

$$\log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.)) \approx \frac{f'(0)}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\alpha^2}{6} - \frac{\alpha f'(0)}{4} + \frac{f''(0)}{4} \right) \langle \mathbf{d}^3 \rangle \quad (17)$$

Törnqvist, Fisher and Diewert Indices

From expression (17) we directly obtain the approximation for the Törnqvist index by setting $\alpha = 0$:

$$\log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{f'(0)}{2} \langle \mathbf{d}^2 \rangle + \frac{f''(0)}{4} \langle \mathbf{d}^3 \rangle \quad (18)$$

By setting $\alpha = 1/2$ we get the approximation for the Diewert index:

$$\log D(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{f'(0)}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{f''(0)}{4} + \frac{1}{24} - \frac{f'(0)}{8} \right) \langle \mathbf{d}^3 \rangle \quad (19)$$

Accordingly, by setting $\alpha = 1$ we obtain the approximation for the Fisher index:

$$\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{f'(0)}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{f''(0)}{4} + \frac{1}{6} - \frac{f'(0)}{4} \right) \langle \mathbf{d}^3 \rangle \quad (20)$$

Expressing these indices relative to the Törnqvist index, we obtain:

$$\log GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{\alpha^2}{6} - \frac{\alpha f'(0)}{4} \right) \langle \mathbf{d}^3 \rangle \quad (21)$$

Thus, for the Diewert and Fisher indices, the difference becomes, respectively:

$$\log D(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{1}{24} - \frac{f'(0)}{8} \right) \langle \mathbf{d}^3 \rangle \quad (22)$$

$$\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{1}{6} - \frac{f'(0)}{4} \right) \langle \mathbf{d}^3 \rangle \quad (23)$$

Therefore, the expressions (18) – (23) will be valid for any smooth weight function on d_i for which the first derivative at $d_i = 0$ exists, for example, $f(d_i) = 1 + \beta d_i$, $f(d_i) = e^{\beta d_i}$, $f(d_i) = 1 + \beta d_i^2$, $f(d_i) = 1 + \log(1 + \beta d_i)$, etc.

Thus, for instance, for function $f(d_i) = 1 + \beta d_i^2$, the first derivative $f'(0) = 0$, and $\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{1}{6} \langle \mathbf{d}^3 \rangle$, and the differences for the weighted indices behave in the same way as the differences for the unweighted (elementary) indices (compare to expression (11)).

Walsh and Lloyd-Moulton indices

The Walsh index is more complicated to derive as it is not part of the $GWE(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, (.))$ family. Using the same methods as above (and omitting some intermediate steps) we get:

$$\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, (.)) \approx \frac{f'(0)}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{1}{24} - \frac{f'(0)^2}{8} + \frac{f''(0)}{4} \right) \langle \mathbf{d}^3 \rangle \quad (24)$$

$$\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, (.)) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, (.)) \approx \left(\frac{1}{24} - \frac{f'(0)^2}{8} \right) \langle \mathbf{d}^3 \rangle \quad (25)$$

Thus, for example, the distance between the Walsh and Diewert indices becomes:

$$\log W(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, (.)) - \log D(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, (.)) \approx \left(\frac{f'(0)}{8} - \frac{f'(0)^2}{8} \right) \langle \mathbf{d}^3 \rangle$$

The Lloyd-Moulton-Törnqvist - Tornqvist distance is straightforward to approximate. From the above (expression (6)) we know that: $\log LM(\mathbf{x}, \mathbf{s}_0, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \frac{\beta^2}{6} \langle \mathbf{d}^3 \rangle$.

Hence,

$$\log LM(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, (.)) \approx \left(\frac{\beta}{2} - \frac{f'(0)}{2}\right) \langle \mathbf{d}^2 \rangle + \left(\frac{\beta^2}{6} - \frac{f''(0)}{4}\right) \langle \mathbf{d}^3 \rangle \quad (26)$$

For example, if $f(d_i) = 1 + \beta d_i$, we get $\log LM(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \frac{\beta^2}{6} \langle \mathbf{d}^3 \rangle$ vs. $(-\frac{\beta^2}{12}) \langle \mathbf{d}^3 \rangle$, if $f(d_i) = e^{\beta d_i}$.

Implicit Price Indices

Implicit price indices are the indices obtained as the ratio of value and volume indices: $P = V / Q$.

Consider one more normalization without restricting generality: $Q(0) = 1$, hence $V(0) = 1$ and $q_i^0 = s_{0i}$. Now, consider the following transformations:

$$x_i = \frac{p_i^1}{p_i^0} / \prod \left(\frac{p_i^1}{p_i^0} \right)^{s_{0i}} = e^{d_i} / e^{\sum s_{0i} d_i} \text{ and } y_i = \frac{q_i^1}{q_i^0} / q = e^{(\beta-1)d_i} / q,$$

where $\prod \left(\frac{p_i^1}{p_i^0} \right)^{s_{0i}}$ is the Geometric Laspeyres price index and q is a volume (quantity) index. For the quantity index q , the exact formula is not important as it gets cancelled out in the derivations below. For example, $q = \prod \left(\frac{q_i^1}{q_i^0} \right)^{s_{0i}}$ (the Geometric Laspeyres volume index) can be implied for the convenience.

Thus, we can write:

$$\begin{aligned} V(\mathbf{p}^1, \mathbf{q}^1, \mathbf{p}^0, \mathbf{q}^0) &= V(1)/V(0) = \\ &= \frac{\sum p_{1i} q_{1i}}{\sum p_{0i} q_{0i}} = q \sum s_{0i} x_i y_i = q \sum s_{0i} e^{d_i} e^{(\beta-1)d_i} = q \sum s_{0i} e^{\beta d_i} \end{aligned}$$

The following functional form describes the implicit price index corresponding to expression (1) for the direct price index $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha)$:

$$iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \sum s_{0i} x_i y_i / \left(\sum s_{0i} y_i^\alpha / \sum s_{1i} y_i^{-\alpha} \right)^{\frac{1}{2\alpha}} \quad (27)$$

Now, noting that $s_{1i} = s_{0i} e^{\beta d_i} / \sum s_{0i} e^{\beta d_i}$, and cancelling out the quantity index q from both the numerator and denominator we get:

$$iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \sum s_{0i} e^{\beta d_i} / \left(\sum s_{0i} e^{\beta d_i} \sum s_{0i} e^{\alpha(\beta-1)d_i} / \sum s_{0i} e^{\beta d_i} e^{-\alpha(\beta-1)d_i} \right)^{\frac{1}{2\alpha}}$$

Using the same methods for approximating functions from above [omitting some intermediate steps and after some algebra], we arrive at:

$$\begin{aligned} \log iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) \\ \approx \frac{\beta^2}{2} \langle \mathbf{d}^2 \rangle + \frac{\beta^3}{6} \langle \mathbf{d}^3 \rangle - \frac{1}{2\alpha} \left(\frac{\beta^2}{2} \langle \mathbf{d}^2 \rangle + \frac{\beta^3}{6} \langle \mathbf{d}^3 \rangle + \alpha^2 \frac{(\beta-1)^2}{2} \langle \mathbf{d}^2 \rangle \right. \\ \left. + \alpha^3 \frac{(\beta-1)^3}{6} \langle \mathbf{d}^3 \rangle - \frac{(\beta+\alpha-\alpha\beta)^2}{2} \langle \mathbf{d}^2 \rangle - \frac{(\beta+\alpha-\alpha\beta)^3}{6} \langle \mathbf{d}^3 \rangle \right) \end{aligned}$$

After collecting the terms, we obtain the following:

$$\begin{aligned} \log iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) \\ \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\alpha^2}{6} + \beta \left(\frac{\alpha}{4} - \frac{\alpha^2}{2} \right) + \beta^2 \left(\frac{\alpha^2}{2} - \frac{\alpha}{2} + \frac{1}{4} \right) + \beta^3 \left(-\frac{\alpha^2}{6} + \frac{\alpha}{4} \right. \right. \\ \left. \left. - \frac{1}{12} \right) \right) \langle \mathbf{d}^3 \rangle \quad (28) \end{aligned}$$

And the difference between the implicit and direct (expression (2)) indices is expressed as:

$$\begin{aligned} \log iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) \\ \approx \left(\beta \left(\frac{\alpha}{2} - \frac{\alpha^2}{2} \right) + \beta^2 \left(\frac{\alpha^2}{2} - \frac{\alpha}{2} \right) + \beta^3 \left(-\frac{\alpha^2}{6} + \frac{\alpha}{4} - \frac{1}{12} \right) \right) \langle \mathbf{d}^3 \rangle \quad (29) \end{aligned}$$

When α equals one, we see that the direct and implicit Fishers are equivalent (this is a well-known fact, of course). For the Tornqvist ($\alpha = 0$), that difference is approximately $-\frac{\beta^3}{12} \langle \mathbf{d}^3 \rangle$. For the Diewert ($\alpha = 1/2$), the difference between the direct and implicit indices is approximately $(-\frac{\beta^2}{8} + \frac{\beta}{8}) \langle \mathbf{d}^3 \rangle$:

$$\log iF(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) = 0 \quad (30)$$

$$\log iD(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log D(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(-\frac{\beta^2}{8} + \frac{\beta}{8} \right) \langle \mathbf{d}^3 \rangle \quad (31)$$

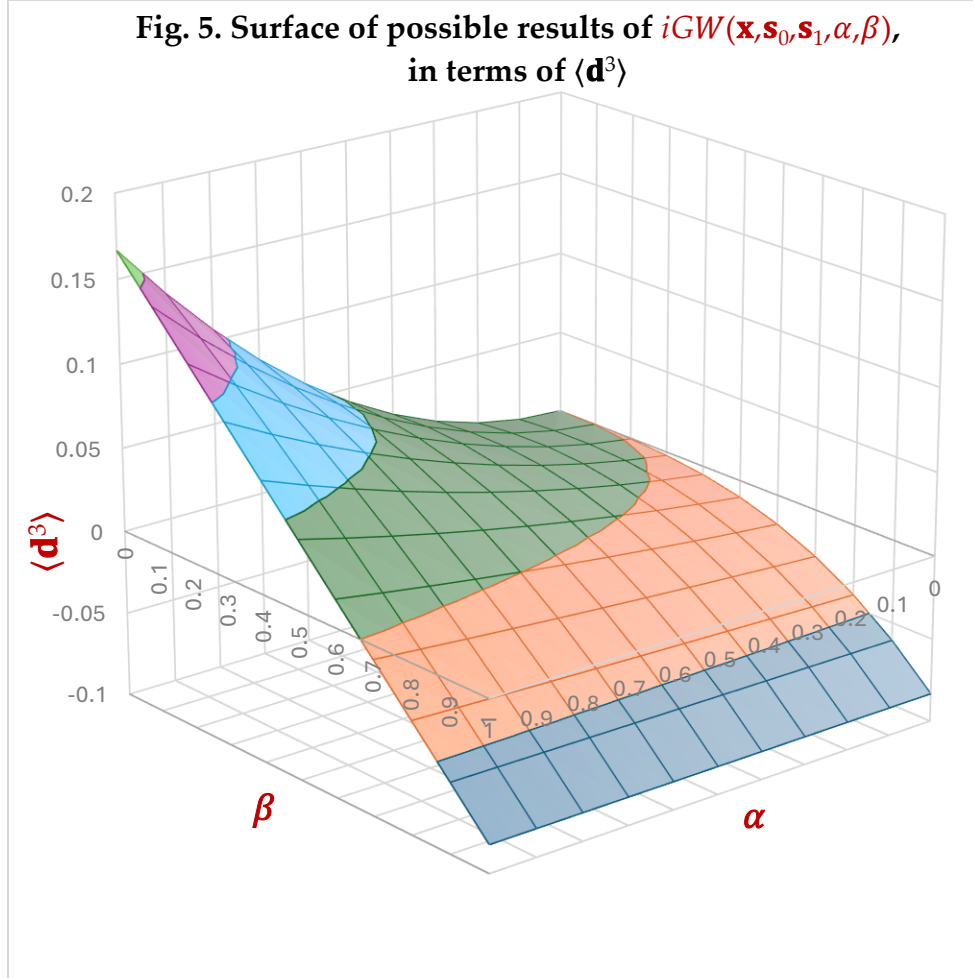
$$\log iT(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx -\frac{\beta^3}{12} \langle \mathbf{d}^3 \rangle \quad (32)$$

This therefore means that the implicit Diewert index is *equivalent* to the Walsh index (see expressions (13) – (14)). And we know that, vice versa, the implicit Walsh is equivalent to the Diewert index. I.e., there exists a *duality* between the Walsh and Diewert indices.

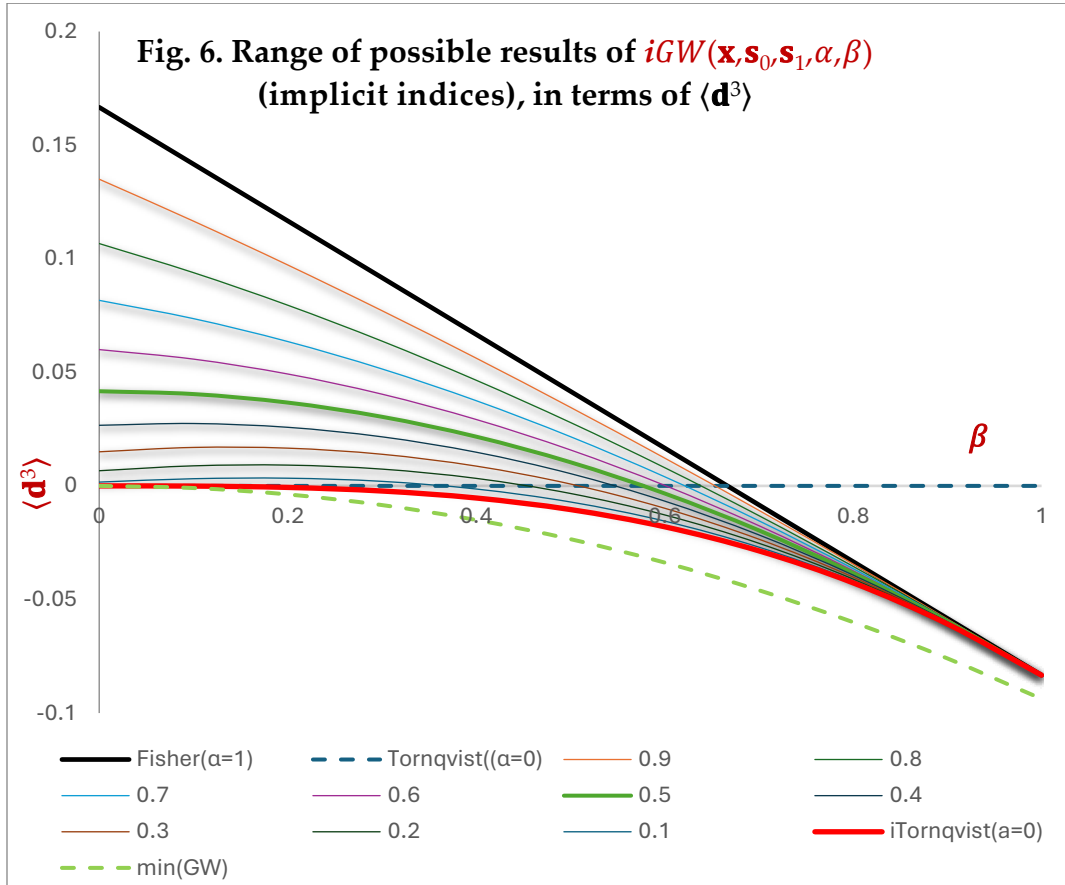
Difference between $iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ and the Tornqvist (direct) index is expressed as:

$$\log iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{\alpha^2}{6} + \beta\left(\frac{\alpha}{4} - \frac{\alpha^2}{2}\right) + \beta^2\left(\frac{\alpha^2}{2} - \frac{\alpha}{2}\right) + \beta^3\left(-\frac{\alpha^2}{6} + \frac{\alpha}{4} - \frac{1}{12}\right)\right)\langle \mathbf{d}^3 \rangle \quad (33)$$

The surface of possible results of expression (33) in the 3D space is presented in Fig. 5 below.



The minimum of $[\log iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta)]$ is achieved at the Törnqvist (implicit) line. The maximum is attained at the Fisher line. Now, consider the projection of this surface onto the $[\langle \mathbf{d}^3 \rangle - \beta]$ plane (Figure 6). The graph shows that the domain of the possible results of $[\log iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta)]$ is bounded by the Fisher index (black line) from above and the implicit Törnqvist index (red line) from below. For comparison, the direct Törnqvist and the minimum of $[\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta)]$ are shown as well.



The implicit Diewert index ($\alpha = 0.5$), which is equivalent to the Walsh index, is shown as the green line.

Note some important properties of the implicit indices: as $\beta \rightarrow 1$, all discussed implicit indices converge to $-\frac{1}{12}\langle \mathbf{d}^3 \rangle$; as $\beta \rightarrow 0$, all discussed implicit indices converge to the corresponding direct indices (see expression (29)).

We see that the range in Figure 6 is tighter than the one in Figure 2 and starting from $\beta = 2/3$ the direct Törnqvist index lies outside the range for the implicit indices. The Fisher, Walsh (implicit Diewert) and implicit Törnqvist are the only indices that are fully located in both ranges. Those three indices can be called truly “ideal”³.

³ Note that all indices described by $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ and $iGW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ are superlative, in the sense that they all approximate each other to the second order at any β .

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ANNEX I

Laspeyres and Paasche indices, PLS

The Laspeyres and Paasche indices play a very important role in index number theory. Using notation from the previous sections, here we derive approximations for the Paasche and Laspeyres indices:

$$\begin{aligned} \log La(\mathbf{x}, \mathbf{s}_0, \beta) &= \log \sum s_{0i} x_i = \log \sum s_{0i} e^{d_i} \approx \log \sum s_{0i} \left(1 + d_i + \frac{d_i^2}{2} + \frac{d_i^3}{6} \right) \\ &\approx \sum s_{0i} \left(\frac{d_i^2}{2} + \frac{d_i^3}{6} \right) \approx \frac{1}{2} \langle \mathbf{d}^2 \rangle + \frac{1}{6} \langle \mathbf{d}^3 \rangle \end{aligned} \quad (34)$$

$$\begin{aligned} \log Pa(\mathbf{x}, \mathbf{s}_1, \beta) &= -\log \sum s_{1i}/x_i = 2\log F(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log La(\mathbf{x}, \mathbf{s}_0, \beta) \\ &\approx \beta \langle \mathbf{d}^2 \rangle + \left(\frac{\beta^2}{2} - \frac{\beta}{2} + \frac{1}{3} \right) \langle \mathbf{d}^3 \rangle - \frac{1}{2} \langle \mathbf{d}^2 \rangle - \frac{1}{6} \langle \mathbf{d}^3 \rangle \\ &= \frac{2\beta - 1}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\beta^2}{2} - \frac{\beta}{2} + \frac{1}{6} \right) \langle \mathbf{d}^3 \rangle \end{aligned} \quad (35)$$

And for the PLS we have the following:

$$\begin{aligned} \log PLS(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) &= \log Pa(\mathbf{x}, \mathbf{s}_1, \beta) - \log La(\mathbf{x}, \mathbf{s}_0, \beta) = \\ &\approx \frac{2\beta - 1}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\beta^2}{2} - \frac{\beta}{2} + \frac{1}{6} \right) \langle \mathbf{d}^3 \rangle - \frac{1}{2} \langle \mathbf{d}^2 \rangle - \frac{1}{6} \langle \mathbf{d}^3 \rangle \\ &= (\beta - 1) \langle \mathbf{d}^2 \rangle + (\beta - 1) \frac{\beta}{2} \langle \mathbf{d}^3 \rangle \end{aligned} \quad (36)$$

For comparison, the geometric PLS will be written as:

$$\log gPLS(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) = 2\log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \beta \langle \mathbf{d}^2 \rangle + \frac{\beta^2}{2} \langle \mathbf{d}^3 \rangle \quad (37)$$

From here, we can see the conditions for $PLS < 1$: when $\langle \mathbf{d}^3 \rangle$ is small (i.e., the skewness of prices is low), $PLS < 1$ holds for all $\beta < 1$ (compare this to the von Bortkiewicz inequality). On the other hand, $gPLS$ is always *greater than one* when $\langle \mathbf{d}^3 \rangle$ is small and $\beta > 0$.

The relationship between the PLS and $gPLS$ can be expressed as:

$$\log PLS(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) - \log gPLS(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx -\langle \mathbf{d}^2 \rangle - \frac{\beta}{2} \langle \mathbf{d}^3 \rangle \quad (38)$$

From here it is easy to derive the condition when the PLS is *greater* than one. If $\beta < 1$, the condition becomes:

$$\langle \mathbf{d}^2 \rangle + \frac{\beta}{2} \langle \mathbf{d}^3 \rangle < 0 \quad (39)$$

At the same time, if $\beta > 0$, expression (39) is the condition of gPLS to be *less* than one (it follows from expression (37)). Thus, when $0 < \beta < 1$ the *log* PLS and *log* gPLS are of opposite signs.

Special case: if $\beta = 0$, i.e., the shares stay constant between the two periods, the Fisher index will be located at $\frac{1}{6}\langle \mathbf{d}^3 \rangle$ above the Törnqvist; and in this case: $\log PLS(\mathbf{x}, \mathbf{s}_0) \approx -\langle \mathbf{d}^2 \rangle$, $\log gPLS(\mathbf{x}, \mathbf{s}_0) = 0$, $\log Pa(\mathbf{x}, \mathbf{s}_0) \approx -\frac{1}{2}\langle \mathbf{d}^2 \rangle + \frac{1}{6}\langle \mathbf{d}^3 \rangle$ and $\log La(\mathbf{x}, \mathbf{s}_0) \approx \frac{1}{2}\langle \mathbf{d}^2 \rangle + \frac{1}{6}\langle \mathbf{d}^3 \rangle$.⁴ Note that $\langle \mathbf{d}^3 \rangle$ can only be a significant value if the prices are skewed substantially.

⁴ Note that all formulae are expressed in terms of $x_i = \frac{p_i^1}{p_i^0} / \prod \left(\frac{p_i^1}{p_i^0} \right)^{s_{0i}}$, i.e., the prices are normalized versus the Geometric Laspeyres index.

ANNEX II

Fifth-order approximation of Generalized Power Index $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$

It is possible to extend expression (2) to include higher-order expansion terms:

$$\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\alpha^2}{6} - \frac{\alpha\beta}{4} + \frac{\beta^2}{4} \right) \langle \mathbf{d}^3 \rangle$$

Using the same methods of derivation, the fifth-term approximation will be written [omitting intermediate steps] as follows:

$$\begin{aligned} \log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) &\approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\alpha^2}{6} - \frac{\alpha\beta}{4} + \frac{\beta^2}{4} \right) \langle \mathbf{d}^3 \rangle + \left(\frac{\alpha^2\beta}{12} - \frac{\alpha\beta^2}{8} + \frac{\beta^3}{12} \right) (\langle \mathbf{d}^4 \rangle + 3\langle \mathbf{d}^2 \rangle^2) \\ &+ \frac{1}{240\alpha} (\alpha^5 - (\beta - \alpha)^5 + \beta^5) (\langle \mathbf{d}^5 \rangle + 10\langle \mathbf{d}^2 \rangle \langle \mathbf{d}^3 \rangle) \quad (40) \end{aligned}$$

An interesting special case when $\beta = 0$ (i.e., item shares kept constant between the two periods) will be written as:

$$\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, 0) \approx \frac{\alpha^2}{6} \langle \mathbf{d}^3 \rangle + \frac{\alpha^4}{120} (\langle \mathbf{d}^5 \rangle + 10\langle \mathbf{d}^2 \rangle \langle \mathbf{d}^3 \rangle) \quad (41)$$

As we can see in this special case the fourth order term disappears.

Thus, expressions (8) - (11) are in fact the fourth order approximations. Hence expression (8), for example, becomes in fact:

$$\log GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, 0) = \frac{\alpha^2}{6} \langle \mathbf{d}^3 \rangle + O(\langle \mathbf{d}^5 \rangle) \quad (8a)$$

The same goes for expressions (9) - (11).

Expressions (40) and (41) are of a purely academic interest, of course, as the third order approximation is quite precise for all practical purposes (see Fig. 4 and the section on precision of the approximations).

ANNEX III

Generalized Weighted Index GW2 that incorporates Walsh index

Generalized index $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$ discussed in the previous sections does not include the Walsh index. However, it is possible to construct an index that would go through the Fisher ($\alpha = 1$) and Tornqvist ($\alpha = 0$) indices in the limiting cases, and, simultaneously, pass through the Walsh index ($\alpha = 1/2$). Let's call it GW2 and define it as follows:

$$GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha) = \left(\sum s_{0i}^\alpha s_{1i}^{1-\alpha} x_i^\alpha / \sum s_{0i}^{1-\alpha} s_{1i}^\alpha x_i^{-\alpha} \right)^{\frac{1}{2\alpha}} \quad (42)$$

Let's assume that item shares change with prices with elasticity β : $s_{1i} = s_{0i} e^{\beta d_i} / \sum s_{0i} e^{\beta d_i}$, where $d_i = \log x_i$. Then, we can rewrite expression (42) as:

$$\begin{aligned} \log GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) &= \frac{1}{2\alpha} \log \left(\sum s_{0i}^\alpha s_{1i}^{1-\alpha} x_i^\alpha / \sum s_{0i}^{1-\alpha} s_{1i}^\alpha x_i^{-\alpha} \right) \\ &= \frac{1}{2\alpha} \left(\log \sum s_{0i} e^{(\alpha+\beta-\alpha\beta)d_i} \right. \\ &\quad \left. - \log \sum s_{0i} e^{(-\alpha+\alpha\beta)d_i} + \log \sum s_{0i} e^{\alpha\beta d_i} - \log \sum s_{0i} e^{(1-\alpha)\beta d_i} \right) \end{aligned}$$

Which, using Taylor expansion as in the case of $GW(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta)$, leads to:

$$\begin{aligned} \log GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) &\approx \frac{1}{2\alpha} \left(\frac{(\alpha\beta)^2}{2} \langle \mathbf{d}^2 \rangle + \frac{(\alpha\beta)^3}{6} \langle \mathbf{d}^3 \rangle - \frac{((1-\alpha)\beta)^2}{2} \langle \mathbf{d}^2 \rangle - \frac{((1-\alpha)\beta)^3}{6} \langle \mathbf{d}^3 \rangle \right. \\ &\quad \left. + \frac{(\alpha + \beta - \alpha\beta)^2}{2} \langle \mathbf{d}^2 \rangle + \frac{(\alpha + \beta - \alpha\beta)^3}{6} \langle \mathbf{d}^3 \rangle - \frac{(-\alpha + \alpha\beta)^2}{2} \langle \mathbf{d}^2 \rangle \right. \\ &\quad \left. - \frac{(-\alpha + \alpha\beta)^3}{6} \langle \mathbf{d}^3 \rangle \right) \end{aligned}$$

Or,

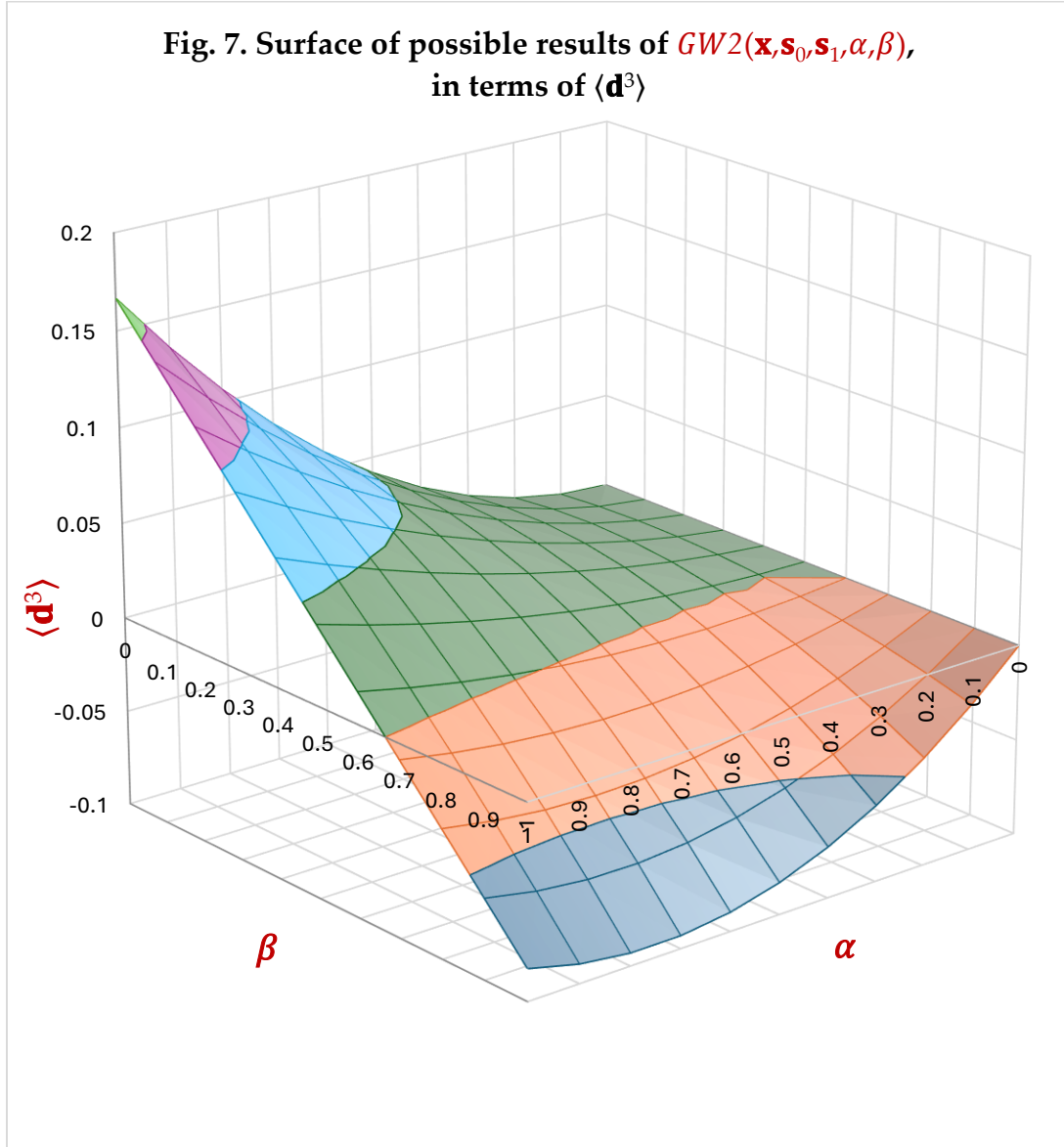
$$\log GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) \approx \frac{\beta}{2} \langle \mathbf{d}^2 \rangle + \left(\frac{\alpha^2}{6} + \beta \left(\frac{\alpha}{4} - \frac{\alpha^2}{2} \right) + \beta^2 \left(\frac{\alpha^2}{2} - \frac{\alpha}{2} + \frac{1}{4} \right) \right) \langle \mathbf{d}^3 \rangle \quad (43)$$

Finally, the difference between GW2 and the Tornqvist index can be expressed as following:

$$\log GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta) \approx \left(\frac{\alpha^2}{6} + \beta \left(\frac{\alpha}{4} - \frac{\alpha^2}{2} \right) + \beta^2 \left(\frac{\alpha^2}{2} - \frac{\alpha}{2} \right) \right) \langle \mathbf{d}^3 \rangle \quad (44)$$

And setting $\alpha = 0, 1/2$ and 1 , we arrive at the Tornqvist (expression (3)), Walsh (expression (7)) and Fisher (expression (5)) indices, respectively.

The surface of possible results of expression (44) in the 3D space is presented in Fig. 7 below.



Note the similarities between Fig. 1 (GW) and Fig. 7 (GW2). Consider the projection of this surface onto the $[\langle \mathbf{d}^3 \rangle - \beta]$ plane (Figure 8 below). The maximum of $[\log GW2(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \alpha, \beta) - \log T(\mathbf{x}, \mathbf{s}_0, \mathbf{s}_1, \beta)]$ is achieved at the Fisher line (until $\beta = 2/3$) and then at the Törnqvist line, just as in the case of GW function. However, the Törnqvist line is also the minimum of the range for GW2 function until $\beta = 1/\sqrt{3}$. Thus, the difference between the ranges for GW and GW2 functions is the area above the yellow line in Fig. 8 ($\min(GW), \alpha = 3\beta/4$) and the area delineated by the Törnqvist line (until $\beta = 1/\sqrt{3}$) and,

after that, by the envelope of lowest boundaries of the lines corresponding to GW2 with $\alpha > \frac{1}{2}$. The Diewert index lies partially outside the range for GW2.

